# Chapter 2

## RESPONSE OF LINEAR CIRCUIT ELEMENTS TO VOLTAGE PULSES

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## CHAPTER 2

## RESPONSE OF LINEAR CIRCUIT ELEMENTS TO VOLTAGE PULSES

### 1. INTRODUCTION

In alternating current theory, it is normal to consider the results of applying sinusoidal voltages to circuits containing resistance, capacitance and inductance. This theory has common application to power and radio-communication systems. In radar and television systems it is equally common to apply to such circuits voltages which are by no means sinusoidal.

When the applied voltage is sinusoidal, the voltages produced across the individual components are also sinusoidal, differing only in phase and magnitude from the input. The relative phases and magnitudes depend on the frequency of the input and on the relative magnitudes of the components.

When the applied voltage is non-sinusoidal the voltages developed across the circuit elements are distorted versions of the input. This makes the circuit behaviour much more complicated, each type of input requiring individual investigation for different relative magnitudes of the circuit components.

This chapter will be devoted to a consideration of the results of applying non-sinusoidal voltages to simple circuits containing linear elements only.

## INSTANTANEOUS APPLICATION OF CHANGES OF VOLTAGE TO CIRCUITS CONTAINING CAPACITANCE AND RESISTANCE

### 2. Instantaneous Application of Change of Voltage To a Series C-R Circuit

Suppose a sudden change of voltage as shown in Fig. 25 is applied to a circuit containing C and R in series. Let the input voltage $v_i$ rise instantaneously from zero to $\hat{v}_i$ at time t = o, and thereafter be maintained indefinitely at this value. Denote the voltage developed across the condenser, of capacitance C, at any instant, by $v_C$ and the voltage across the resistor, of resistance R by $v_R$. Simple theory (Admiralty Handbook of Wireless Telegraphy, BR229 , para. 174, and AP 1093 Part II, Chap. VII, para. 59) shows that the voltage $v_C$ rises exponentially as given by

$$v_C = v_i \, (1 - \varepsilon^{-t/CR})$$

and as illustrated in Fig. 25 . By Kirchhoff's law we know that the sum of the voltages across the capacitance and resistance must at all instants equal the applied voltage, i.e.,

$$v_i = v_R + v_C ,$$

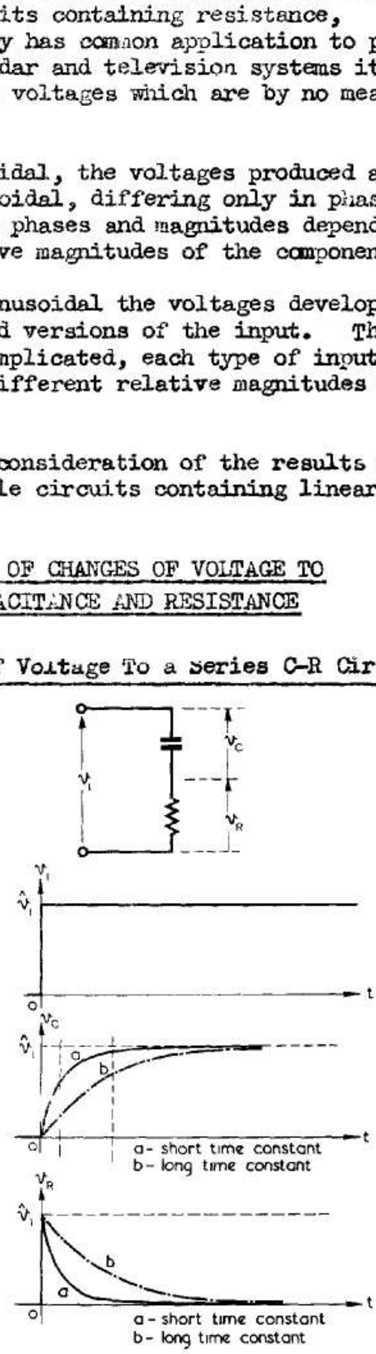


Fig.25.- Response of a C-R circuit to an instantaneous rise of voltage.

so that the voltage $v_R$ is given by

$$v_R = \hat{v}_i \varepsilon^{-t/CR}$$

as shown in Fig. 26

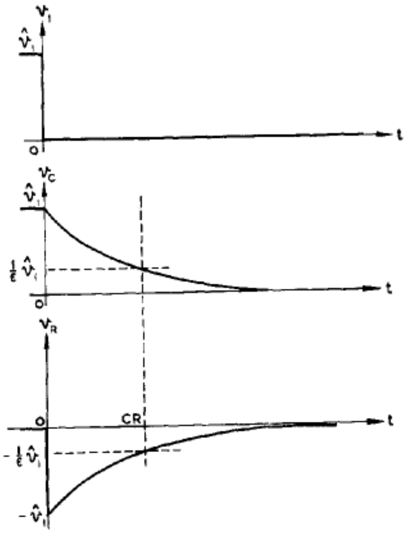


Fig. 26. - Response of a C-R circuit to an instantaneous fall of voltage.

The time-constant CR appropriate to these changes represents the time taken for the voltage across the resistance to fall to $\varepsilon^{-1}\hat{v}_i$ i.e., about one third of its original value. After a time equal to about 5CR has elapsed the voltage has fallen to less than 1% of its original value, and for most practical purposes the change may be considered complete.

If the sudden change of voltage is a decrease instead of an increase similar considerations apply, and the results are as shown in Fig. 26

## 3. Application of a Rectangular Pulse of Voltage to a Series C-R Circuit

We now consider the application of a voltage whose time variation is as shown in Fig. 27 . We shall call this a Rectangular Pulse of Voltage. It is simplest to consider in turn firstly the effect of the sudden increase of voltage at the start of the pulse and secondly the effect of the sudden decrease at the end. We have already dealt with these two cases separately.

Assume first that the time-constant CR of the circuit is considerably less than the time of duration T of the pulse. Then the charging of the condenser is completed (in so far as an exponential rise is ever completed) before the discharge takes place, and the voltages developed across the condenser and resistor are as shown in Fig. 27 . Two sharp narrow pulses of voltage of opposite sign are produced across the resistor, one at the start and one at the end of the applied pulse. The durations of these sharp pulses depend on the magnitude of the time-constant.

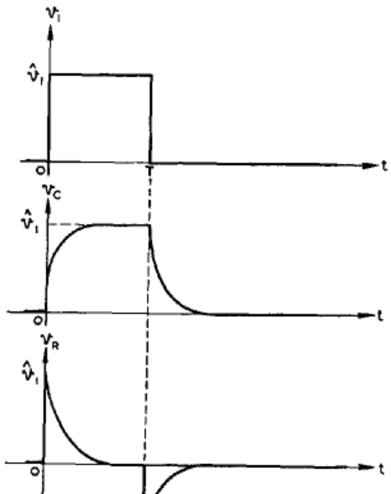


Fig. 27. - Response of a C-R circuit to a rectangular pulse. $CR \ll T$

If the time-constant of the circuit is made variable (usually, for convenience, by changing the value of R) the width of these short duration pulses developed across the resistor can be controlled. In the ideal case considered the full voltage $\hat{v}_i$ would be developed across the resistor however short the time-constant. In practice this is not so for reasons discussed later (Secs. 5 and 12), and the amplitude of $v_R$ tends to be further reduced the smaller the value of the time-constant.

Whilst the duration of the short pulses developed across the resistor depends on the time-constant of the circuit, it does not depend on the duration T of the input rectangular pulses. The time which elapses between the leading edges of the first and second pulses developed across the resistor is equal to T.

Suppose next that the time-constant of the circuit is very much greater than the duration of the applied pulse. In this case the voltage across the condenser rises only slightly before the input voltage drops and causes the condenser to discharge (Fig. 28 ). Consider the case when $CR \doteq 10T$; then at time T (time at which input voltage drops) $v_C$ is given by :-

$$v_C = \hat{v}_i\,(1 - \mathcal{E}^{-t/CR})$$
$$= \hat{v}_i\,(1 - \mathcal{E}^{-1/10})$$
$$= \hat{v}_i\,(1 - 0{\cdot}905)$$
$$= 0{\cdot}095\hat{v}_i$$

After time T the condenser discharges from the value $0{\cdot}095\hat{v}_i$ towards zero. The value of $v_C$ after time 2T (time T after the instant at which the input voltage falls to zero) is given by :-

$$v_C = 0{\cdot}095\hat{v}_i\,\mathcal{E}^{-t/CR} \quad (0{\cdot}095\hat{v}_i = \text{potential difference at start of discharge})$$
$$= 0{\cdot}095\hat{v}_i\,\mathcal{E}^{-1/10}$$
$$= 0{\cdot}095\hat{v}_i \times 0{\cdot}905$$
$$= 0{\cdot}086\hat{v}_i$$

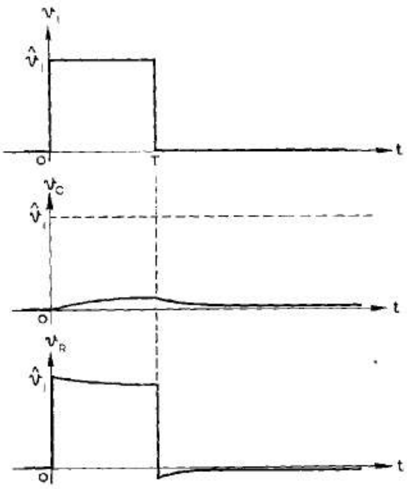


Fig.28.- Response of a C-R circuit to a rectangular pulse. $CR \gg T$.

This difference between the rates of charging and of discharging of the condenser under the above conditions can be summed up as follows. The rate of charge or discharge is proportional to the voltage applied to the resistor. At the beginning of the charging period this voltage is $\hat{v}_i$, whilst at the beginning of the discharging period it is only $0{\cdot}095\hat{v}_i$. Thus the rate of discharge is less than one-tenth the rate of charge. The voltage $v_R$ across the resistor rises to a value $\hat{v}_i$ at the onset of the rectangular input pulse, falls, at the same rate as $v_C$ rises, for the duration T, and then drops instantaneously by an amount $\hat{v}_i$ with the end of the input pulse. Finally $v_R$ rises exponentially towards zero at the same rate as $v_C$ falls. The longer the time-constant CR, the more faithfully does $v_R$ reproduce the input voltage. Thus when $CR = 100T$ the drop in $v_R$ throughout the duration T of the input pulse is only $0{\cdot}01\hat{v}_i$ and the output pulse across the resistor is very little distorted compared with the input.

In radar systems it is common practice to apply a rectangular pulse to a C-R circuit such as that of Fig. 28 and to utilise the voltage developed across the resistor. The above considerations reveal the following points :-

(i) The maximum value of the voltage developed across R is in all cases equal to the magnitude of the applied pulse.

(ii) The shape of the output voltage is a close replica of the input if $CR \gg T$; but if $CR \ll T$ the input rectangular pulse is converted into two narrow pulses of opposite sense to each other, whose durations depend on the value of the time-constant CR.

## 4. Application of a Succession of Rectangular Pulses of Voltage To a Series C-R Circuit

We now consider the case where a succession of rectangular pulses of voltage is applied to the circuit of Fig. 25 . Suppose $T_1$ is the duration of each pulse and $T_2$ the interval between the pulses.

The general shapes of the voltages across C and across R are similar to those already described. However, the mean level of the voltage across the condenser tends to rise with each successive pulse; unless the time-constant of the circuit is so short compared with either $T_2$ or $T_1$ that the condenser can be considered to have discharged or charged completely during these periods.

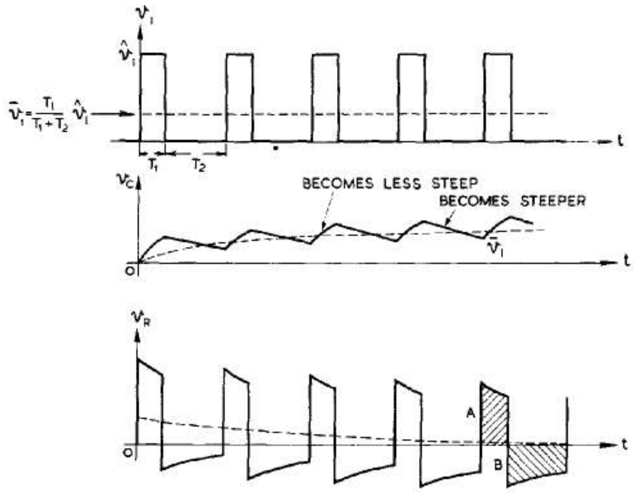


Fig.29.- Application of a succession of rectangular pulses to a C-R circuit. $CR \gg T_1$

Ultimately a Steady State is reached in which the charge acquired by the condenser during the time $T_1$ is equal to the charge lost during the time $T_2$. This means that the mean current flowing into the condenser through the resistor is zero when equilibrium has been reached, so that the mean value of $v_R$ is also zero. The graph showing the voltage developed across the resistor is such that the area (voltage x time) enclosed above the zero (mean) voltage line during the time $T_1$, is equal to the area enclosed below this line during the time $T_2$, i.e., in the steady state the shaded areas A and B (Fig. 29 ) are equal.

Alternatively, a useful way of approaching the problem is to consider the succession of positive input pulses to be composed of a steady component of voltage and a purely alternating component of voltage.

The mean value $\bar{v}_i$ of the input voltage is given by :-

$$\bar{v}_i = \frac{T_1}{T_1 + T_2} \hat{v}_i$$

The alternating component of course has a mean value of zero. If a steady voltage is applied to a condenser and resistor in series, the whole of the voltage appears ultimately across the condenser and, as we have just seen, the rise is exponential. In the present case therefore the mean voltage across the condenser increases exponentially with time-constant CR towards the value

$$\frac{T_1}{T_1 + T_2} \hat{v}_i$$

Ultimately the mean voltage across the resistance is zero.

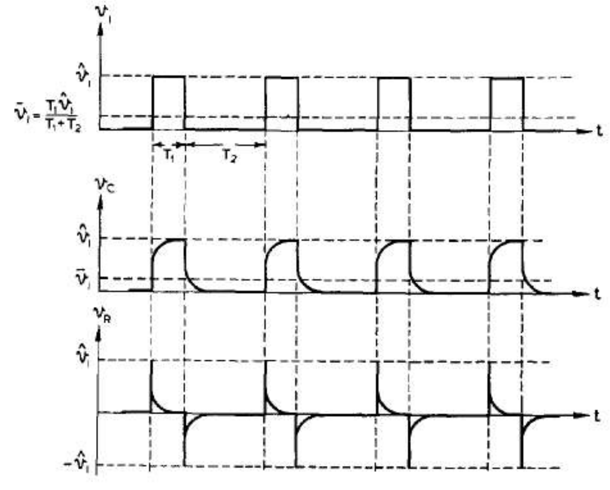


Fig. 30. - Application of a succession of rectangular pulses to a C-R circuit. $CR \ll T_1$

It is therefore apparent that if $CR \gg T_1$ and $T_1 < T_2$ the ultimate output voltage across R is a replica of the input voltage across C in so far as the alternating portion of the input is concerned; but that the steady component of the input voltage variation is not reproduced. If it is required to reproduce this steady component, which is determined by the mean voltage level of the input, special measures have to be employed. This process of restoring the steady (or DC) component of voltage is often called DC Restoration (See Chap. 12 Sect. 2).

If $T_2 < T_1$ the input can be considered as a succession of negative-going pulses, and a similar argument holds. In this case the condition for faithful reproduction of the alternating component of the applied pulse in the voltage across the resistor is that $CR \gg T_2$.

If CR is short compared with both $T_1$ and $T_2$ the steady state is rapidly approached, and this is illustrated in Fig. 30.

## 5. Instantaneous Application of a Change of Voltage to Circuits Containing More Than One Condenser or Resistor.

Fig. 31 shows a circuit consisting of two resistors $R_1$ and $R_2$ in series with a condenser C. The sum of the voltages developed across $R_1$ and $R_2$ behaves exactly the same way as the voltage $V_R$ across R in the circuit of Sec. 1-4, with

$$R = R_1 + R_2$$

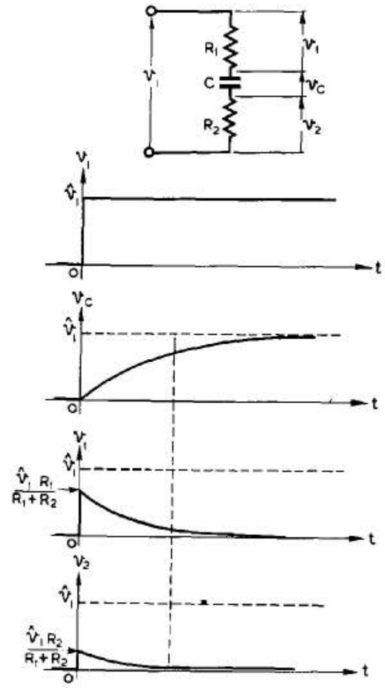


Fig. 31. - Response of a C-R circuit which contains more than one resistor.

The voltage developed across each resistor is proportional to its resistance, i.e.,

$$v_1 = \frac{R_1}{R_1 + R_2} v_R \text{ and } v_2 = \frac{R_2}{R_1 + R_2} v_R$$

The time-constant is

$$C (R_1 + R_2)$$

A circuit commonly encountered in radar is shown in Fig. 32 . $R_1$ represents the output resistance of one amplifier, whilst $C_2$ is the input capacitance of the next stage. In general $C_1 \gg C_2$. Unfortunately the general analysis of this circuit is too complicated for inclusion here, but if the output resistance of the pulse generator is made very small the circuit of Fig. 33 may be taken as a sufficiently close approximation.

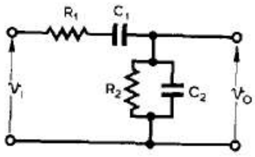

Fig. 32. - Practical form of C-R circuit.

At time t = 0 the input voltage rises almost instantaneously from zero to $\hat{v}_i$. (This rise is actually exponential, but because the generator output resistance is very small, the effective time-constant is negligible). This rise of voltage is developed across the two condensers in the inverse ratio of their capacitances. While the input voltage is maintained at $\hat{v}_i$, the condenser $C_1$ charges exponentially so that the voltage across it rises towards the value $\hat{v}_i$, and $C_2$ discharges exponentially towards zero. The time-constant of these variations can be shown to be $R_2 (C_1 + C_2)$.

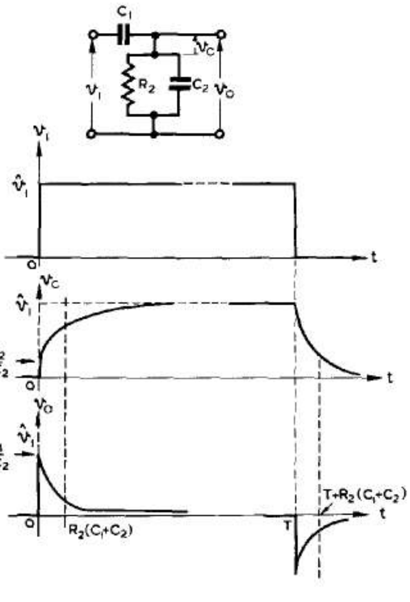

Fig. 33. - Response of practical C-R circuit (simplified).

It will be noticed that one effect of the condenser $C_2$ is to reduce the maximum value of the output voltage developed across the resistor $R_2$. Thus, if the condenser $C_1$ is reduced in value in order that the time-constant may be small, i.e., so that the output voltage may consist of short duration pulses, the amplitude of these pulses is reduced.

## INSTANTANEOUS APPLICATION OF CHANGES OF VOLTAGE TO CIRCUITS CONTAINING INDUCTANCE AND RESISTANCE

### 6. Instantaneous Application of a Change of Voltage to a Series L-R Circuit (Fig. 34 ).

Let the input voltage $v_i$ rise instantaneously from zero to $\hat{v}_i$ at time t = 0 and thereafter be maintained indefinitely at this value. Denote by $v_L$ the voltage developed at any instant across the coil, of inductance L, and by $v_R$ the voltage across the resistor, of resistance R.

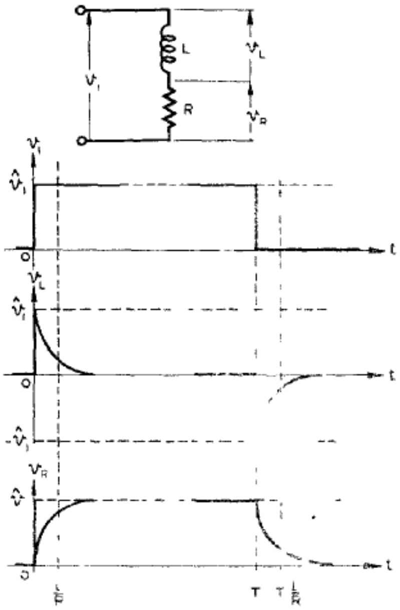


Fig. 34. - Response of L-R circuit to rectangular pulse. $L/R \ll T$

Standard theory (BR229 Admiralty Handbook of Wireless Telegraphy, Vol. I, para. 158, and AP.1093, Part II, Chap. II, para. 43) shows that the rise in current in the circuit is given by the expression :-

$$i = \frac{\hat{v}_i}{R}(1 - \varepsilon^{\frac{-Rt}{L}})$$

so that the voltage developed across the resistance R is

$$v_R = \hat{v}_i \ (1 - \varepsilon^{\frac{-Rt}{L}})$$

At any instant $v_L$ may be determined by the relation

$$v_i = v_R + v_L$$

The time-constant is L/R.

Fig. 34 also shows the effect of reducing, instantaneously, the input voltage from a value $\hat{v}_i$ to zero, assuming that the voltage across the coil and resistor have attained their final values before the change takes place. The full drop of voltage appears instantaneously across the coil. Subsequently the voltage across the resistor falls exponentially from $\hat{v}_i$ towards zero, while that across the coil rises exponentially from a $-\hat{v}_i$ towards zero.

It should be noted that the nature of the voltage variation across the coil is similar to that across the resistor in the corresponding C-R circuit, whilst the voltage variation across the resistor in the present case is similar to that developed across the condenser.

## 7. Application of a Rectangular Pulse of Voltage to a Series L-R Circuit

Consideration similar to those of Sect.3 apply to a circuit consisting of a coil L and resistor R in series. We shall consider separately the two cases (i) $\frac{L}{R} \ll T$ and (ii) $\frac{L}{R} \gg T$, T being the duration of the input pulse.

(i) $\frac{L}{R} \ll T$ (Fig. 34)

The voltage across the resistor rises exponentially to practically its full value $\hat{v}_i$ during the interval T, while the voltage across the coil, after its initial instantaneous rise to $\hat{v}_i$, falls exponentially to zero. At time T the input voltage drops instantaneously to zero, so the voltage across the coil drops instantaneously to $-\hat{v}_i$. After time T the voltage across the coil rises exponentially from $-\hat{v}_i$ towards zero, while the voltage across the resistor falls exponentially from $\hat{v}_i$ towards zero.

Thus voltage pulses of short duration are developed across the coil.

(ii) $\frac{L}{R} \gg T$

The voltage across the resistor rises exponentially by only a small amount before the end of the applied pulse; then the voltage across the resistor starts to fall exponentially. This behaviour is similar to that of $v_C$ in Fig. 28. The voltage across the coil rises instantaneously from zero to a value $\hat{v}_i$ at time $t = 0$, falls exponentially by a small amount during the interval $t = 0$ to $t = T$, and then drops instantaneously by an amount $\hat{v}_i$. Subsequently the voltage developed across the coil rises exponentially towards zero. The voltage developed across the coil is an almost undistorted reproduction of the input pulse.

## 8. Application Of a Succession Of Rectangular Pulses of Voltage to a Series L-C Circuit

As in the corresponding case of the C-R circuit (Fig. 29, Sect.4), the mean value of the input is given by :-

$$\bar{v}_i = \frac{T_1}{T_1 + T_2} \hat{v}_i$$

The series of pulses may be split up into a steady component equal to this mean value, and an alternating component whose mean value is zero. The steady component of voltage across the resistor ultimately attains the mean value $\bar{v}_i$, and approaches this value at a rate dependent upon the time-constant of the circuit. In the steady state only the alternating component, which represents the rectangular pulse shape, appears across the coil.

## 9. DISADVANTAGES OF A SERIES L-R CIRCUIT COMPARED WITH A SERIES C-R CIRCUIT

It is not usually convenient to employ L-R networks to provide time-constants of more than a few microseconds such as are commonly required in radar. Such a network is usually in series with some form of generator whose output resistance is of the order of a few thousand ohms, so that the size of coil required to give the desired $\frac{L}{R}$ ratio is inconveniently large. The requirements can be met by the use of C-R components of more practicable values. Further, the inherent resistance of a condenser is usually negligible whereas that of a coil is always appreciable. Even when very short time-constant circuits are required, coils cannot be used to produce the effects described above because of their self-capacitance. Their use in ringing circuits, where self-capacitance is not necessarily deleterious, is described in Sec. 10.

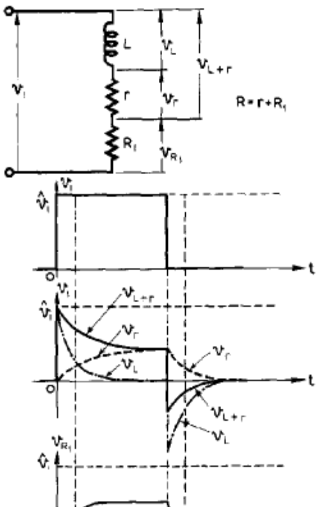


Fig.35.- Modifications to response of L-R circuit due to coil resistance.

Fig. 35 shows the effect of coil resistance on the voltages across coil and resistor in a series L-R circuit. The voltage developed across the resistor is affected only in magnitude, but the coil voltage may be considerably modified as shown (compare $v_{L+r}$ of Fig. 35 with $v_L$ of Fig. 34).

## INSTANTANEOUS APPLICATION OF CHANGES OF VOLTAGE TO CIRCUITS CONTAINING INDUCTANCE, CAPACITANCE AND RESISTANCE

### 10. Instantaneous Application of a Change of Voltage to An L-C-R Circuit

The consideration of free oscillations in a series L-C-R circuit (Fig.36(a) is dealt with in many standard works (Admiralty Handbook of Wireless Telegraphy, BR229 , para. 390, and AP.1093, Part II, Chap. VII, paras. 7-16). It is shown that the sudden application of a change of voltage to this kind of circuit results either in a free oscillation or in a smooth, non-oscillatory change, according to the relative values of the components.

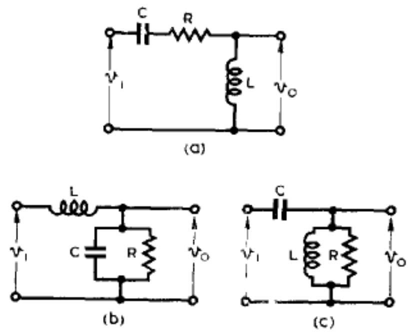


Fig.36. - L-C-R circuits.

In the case of a free oscillation the frequency depends mainly upon L and C and to a lesser extent upon R, and the amplitude decreases exponentially according to the amount of damping in the circuit.

The quantity $\zeta = \frac{R}{2}\sqrt{\frac{C}{L}}$ is called the Damping Ratio.

If $\zeta < 1$ the Circuit will perform a damped oscillation with a frequency f and a logarithmic decrement $\delta$ given by :-

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

$$\delta = \frac{R}{2f_0L}$$

If the resistance is very small the frequency is approximately $\frac{1}{2\pi\sqrt{LC}}$
Such a circuit is called a Ringing Circuit.

If $\zeta = 1$ the output voltage of the circuit is just not oscillatory (Critical Damping).

If $\zeta > 1$ the output voltage takes the form of an exponential rise.

In many radar applications it is desirable to use a ringing circuit whose damping is slightly less than critical.

Ringing circuits are frequently used with the damping resistor R in parallel with either the coil or the condenser instead of in series, as shown in Figs.36(b)and(c).In these cases it can be shown that, neglecting the resistance of the coil, critical damping occurs when $R = \frac{1}{2}\sqrt{\frac{L}{C}}$, and that oscillations take place for values of R greater than this.

Fig. 37 shows the output voltage produced across the condenser in the circuit of Fig.36(b) when a change of voltage is instantaneously applied at the input terminals. Diagrams illustrate the cases $R = \sqrt{\frac{L}{C}}$ (damped oscillatory), $R = \frac{1}{2}\sqrt{\frac{L}{C}}$ (critically damped) and $R = \frac{1}{4}\sqrt{\frac{L}{C}}$ (non-oscillatory).
In all cases the delay-time before the output voltage reaches the value

of the input is proportional to $\sqrt{LC}$. The voltage across the coil may be obtained from the relation $v_i = v_L + v_C$.

## 11. INSTANTANEOUS APPLICATION OF A CHANGE OF VOLTAGE TO A DELAY NETWORK

A more detailed consideration of this problem is given in Chap. 4, where the wave-nature of the effect is dealt with. As an introduction, and for the sake of completeness, a brief mention is made here, considering the behaviour of the delay network as an extension of that of the ringing circuit.

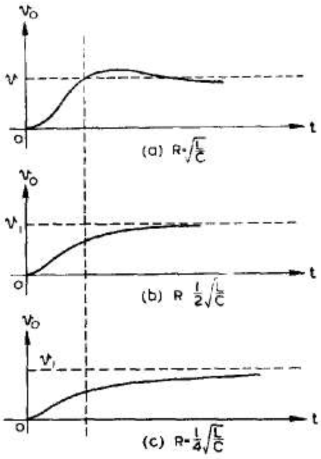


Fig.37.- Response of L-C-R circuit of fig.36(b).

Fig. 38 shows two-stage and six-stage networks composed of identical sections, such as are used to make up a pulse-forming Delay Line. In Sec. 10, illustrated by Figs.36 & 37 , we have already discussed the behaviour of the single-stage network, or ringing circuit. If we replace the resistance R of Fig.36 (b) by another ringing circuit identical with the original one, we have the circuit of Fig. 38(a). We shall assume that this does not appreciably affect the voltage $v_1$ produced across the first condenser when the input voltage is applied. This voltage is shown in Fig.38(c) . $v_1$ is in turn applied to the ringing circuit formed by the second L-C section and R. The effect produced is not vastly different from that which would occur if the input pulse were applied to this network, not at time $t = 0$, but at time $t = T$. In other words, the voltage $v_o$ across R is similar to $v_1$ except that $v_o$ does not reach the value of the input voltage until $t = T'$, where $T'$ is approximately equal to $2T$. Subsidiary effects also occur, such as minor alterations in the character of the ringing at the end of the delay time $T'$. (Fig. 38(d) ).

The addition of further L-C sections produces a similar effect. The delay time before the rise in input voltage appears across the output is approximately proportional to the number of stages used. Neither the amplitude of the ringing, nor its duration, is appreciably affected by an increase in the number of L-C sections.

In the simple ringing circuit of Fig. 36(b) the delay T and the period of the ringing are both proportional to $\sqrt{LC}$. In the multiple-stage network the total delay is proportional to $n\sqrt{LC}$, where n is the number of stages, whilst the period of the ringing is approximately independent of n. By increasing n and reducing L and C proportionately, the ringing can be made negligible without increasing the delay time. If this is done in a six-stage or eight-stage network, the output voltage is made approximately rectangular, as shown in Fig. 38(e).

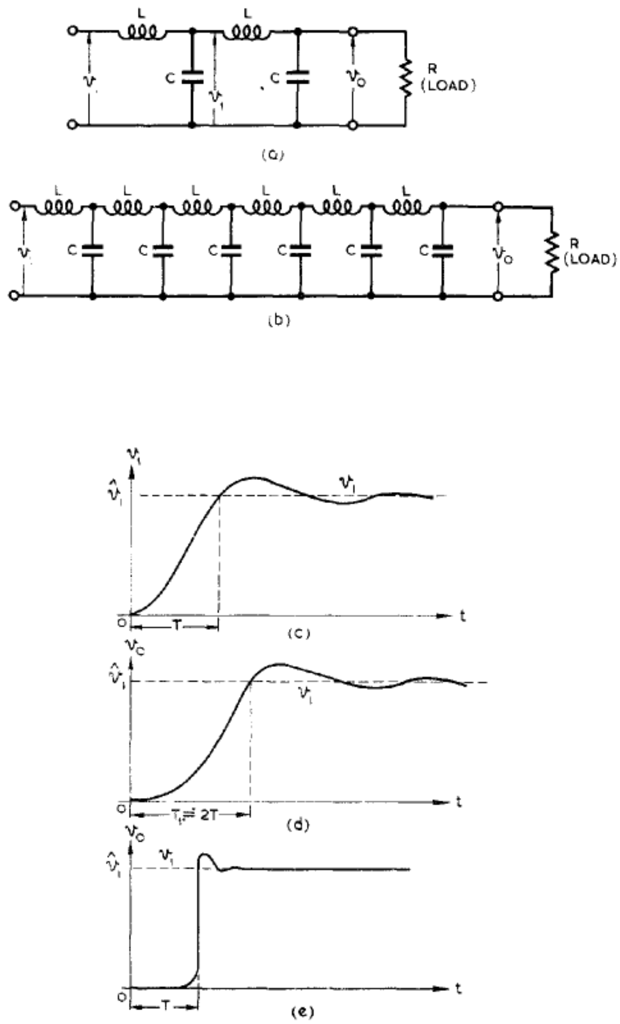


Fig. 38. - Response of delay network.

NON INSTANTANEOUS CHANGES OF VOLTAGE

## 12. General Discussion

In practice it is impossible to produce an absolutely instantaneous change in voltage, but in many cases no great error is introduced in considering a change of voltage to be instantaneous if the total time taken for the change is short compared with 0.1 /us. However, sometimes the change of voltage applied across a circuit must be considered as changing linearly with time. A common example is the sawtooth voltage produced by time-base circuits (Fig. 42 ). Sometimes the rise or fall of voltage is exponential, but usually only a small portion of the exponential rise or fall is considered and for practical purposes this portion may be regarded as linear.

## 13. Input Voltages Increasing at a Constant Rate

Let the input voltage $v_i$ applied to the condenser (capacitance C) and resistor (resistance R) in series increase at a constant and finite rate m, so that

$$v_i = mt$$

The voltage $V_R$ across the resistor rises exponentially and tends to a final value mCR. The initial rate of rise is the same as that of the input voltage $v_i$.

The voltage $v_C$ across the condenser also begins to rise exponentially, but tends to rise at the same rate as $v_i$ when $v_R$ approaches its steady value. This is indicated in Fig. 39.

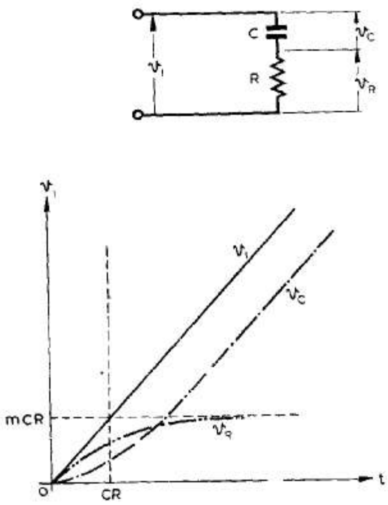


Fig. 39. - Response of C-R circuit to voltage varying linearly with time.

Mathematical analysis

At time t let :-

q = charge on condenser

i = current in circuit $\left(= \frac{dq}{dt}\right)$

Then :-

applied voltage = voltage across condenser + voltage across resistor

or $v_i = mt = \frac{q}{c} + iR$

Differentiating with respect to t :-

$$m = \frac{1}{C}\frac{dq}{dt} + R\frac{di}{dt}$$

or $$m = \frac{i}{C} + R\frac{di}{dt}$$

The solution of this equation is :-

$$i = mC\ (1 - \varepsilon^{-t/CR})$$

Therefore $v_R = iR = mCR\ (1 - \varepsilon^{-t/CR})$

and $v_C = v_i - v_R = mt - mCR(1 - \varepsilon^{-t/CR})$

Thus $v_R$ rises exponentially to a limiting value mCR. The smaller is the time-constant CR or the slower the rate of rise of the input voltage (smaller m), the lower does this final value of $v_R$ become. The rate of rise of $v_R$ is

$$\frac{d}{dt}(v_R) = m\,\varepsilon^{-t/CR}$$

so that at time t = 0 the rate of rise is m, i.e., $v_R$ initially rises at the same rate as $v_i$. The rate of rise of voltage across the condenser starts from zero and increases until, by the time $v_R$ is constant at a value mCR, it becomes the same as that of the input voltage.

If a voltage which rises linearly with time is applied to a coil and resistor in series, the voltage $v_L$ across the coil and the voltage $v_R$ across the resistor at any instant are given by the expressions

$$v_L = m\,\frac{L}{R}(1 - \varepsilon^{-\frac{Rt}{L}}),$$

$$\text{and} \quad v_R = mt - \frac{mL}{R}(1 - \mathcal{E}^{\frac{-Rt}{L}})$$

The voltage across the coil rises towards a final value $\frac{mL}{R}$, and is similar to the voltage across the resistor in the corresponding C-R circuit. The voltage across the resistor in the L-R circuit is similar to that across the condenser in the C-R arrangement.

## 14. Linear Rise of Voltage of Finite Duration

Now let us suppose that the input voltage applied to a series C-R circuit rises at a rate m for a time T' and then remains constant at a value mT' for an indefinite time; (Fig. 40 ).

During the time T' the voltage across the resistor rises exponentially towards a maximum value equal to mCR. Throughout this time $v_R$ is given by :-

$$v_R = mCR\ (1 - \mathcal{E}^{-t/CR})$$

During the time the input voltage remains constant $v_R$ falls exponentially from its value at T' towards a final value of zero.

Three cases arise, in which CR $\gtreqless$ T'. These are illustrated by the following examples corresponding to the three diagrams of Fig. 40
We are concerned here only with the value of $v_R$ during the rise of $v_i$.

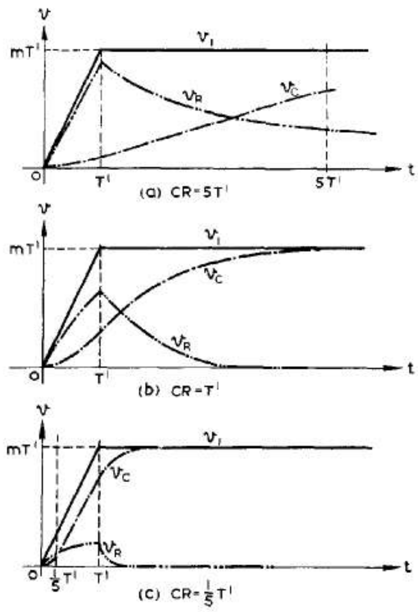


Fig.40. - Response of C-R circuit to linear voltage rise of finite duration.

(i) CR = 5T'

At time T' the value of $v_R$ is 0·9 mT', i.e., the voltage across the resistor rises to nine-tenths of the maximum amplitude of the applied voltage. The greater the time-constant, the more closely $v_R$ approaches the amplitude of the input voltage.

(ii) CR = T'

The value of $v_R$ at the time T' is approximately two-thirds of the maximum amplitude of the input voltage.

(iii) CR = $\frac{1}{5}$ T'

The value of $v_R$ at the time T' is nearly equal to one-fifth of the maximum amplitude of the input voltage, i.e., the smaller the time-constant the smaller is the value of $v_R$.

The voltage across the condenser at any instant may be obtained from the relation

$$v_i = v_C + v_R$$

This is illustrated in Fig. 40 .

It should be noted that since there is no instantaneous change in the applied voltage at any time, in particular at t = 0 and t = T', there is never any sudden change in the rate of rise of $v_C$.

## 15. Pulses Involving Linear Variations of Voltage

Fig. 41 shows the voltage developed across the resistor, and that developed across the condenser, when a trapezium-shaped pulse is applied. A so-called "rectangular" pulse usually has this trapezium shape, i.e., the changes of voltage are not instantaneous. It has been assumed that the input voltage rises to its constant value in an interval T', remains at this value for a time T, and then falls back to its initial value in a further interval T'. T must be very much greater than T' if the voltage change is to represent the usual type of "rectangular" pulse. When the time-constant of the circuit is smaller than T, short pulses are produced across the resistor (compare Sec. 3). However, if the time-constant is made progressively smaller, the amplitude of these short pulses decreases. When the time-constant is much greater than the time T, the voltage developed across the resistor closely approximates to the input voltage.

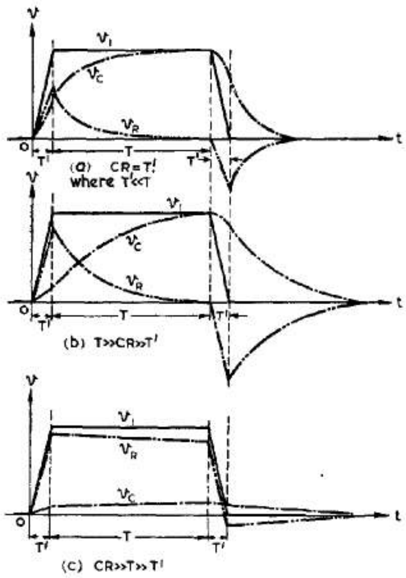


Fig.41. - Response of C-R circuit to trapezoidal pulse.

Fig. 42 shows the voltage developed across the resistor and that developed across the condenser when a pulse of triangular shape is applied. The rise of input voltage takes place in an interval T', and the fall of input voltage is assumed to be instantaneous. Such a variation is commonly called a Sawtooth Voltage. When the time-constant of the circuit is much greater than T', the voltage across the resistor closely resembles the input voltage. When the time-constant is much smaller than T', the voltage variation across the condenser approximates in shape to that of the input voltage, whilst the voltage across the resistor rises to a small value during T' and at t = T' drops sharply to form a negative-going pulse.

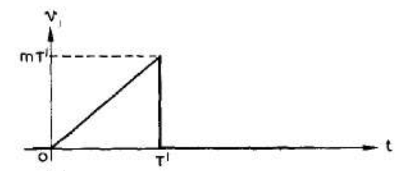


Fig.42. - Response of a C-R circuit to sawtooth pulse.

Fig. 43 shows the voltages developed across the resistor and the condenser when a triangular pulse which has equal rates of rise and fall of voltage is applied to the circuit. It should be noted that, when the time-constant of the circuit is less than the time of rise or fall of the input voltage, the voltage variation developed across the resistor approximates to a pair of rectangular pulses of small amplitude.

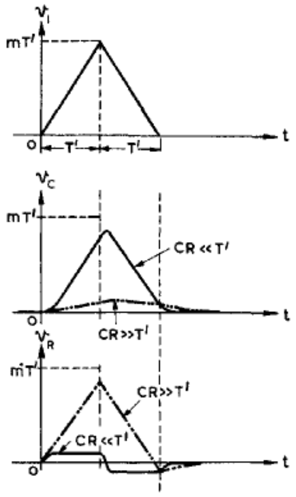


Fig. 43. - Response of a C-R circuit to triangular pulse.

## 16. Succession of Pulses

As in the case of the rectangular pulses dealt with in Secs. 4 and 8, an input which consists of a succession of uniform pulses of any shape may be divided into a steady component equal to the mean value $\bar{v}_i$ of the input and an alternating component whose mean value is zero. When such an input is applied to a series C-R circuit, an equilibrium condition is ultimately arrived at in which the steady component of the voltage across the condenser is equal to $\bar{v}_i$ whereas the mean value of the voltage across the resistor is zero. The fraction of the alternating component appearing across either the condenser or the resistor, and the amount of distortion present, depend on the time-constant.

Fig. 44 shows the waveforms of some pulses which are commonly applied to a series C-R circuit. Fig. 45 shows the voltages developed in the steady state across the resistor and condenser when a continuous train of triangular pulses is applied to the circuit. Fig. 46 shows the results to be expected when a continuous train of sawtooth pulses is applied.

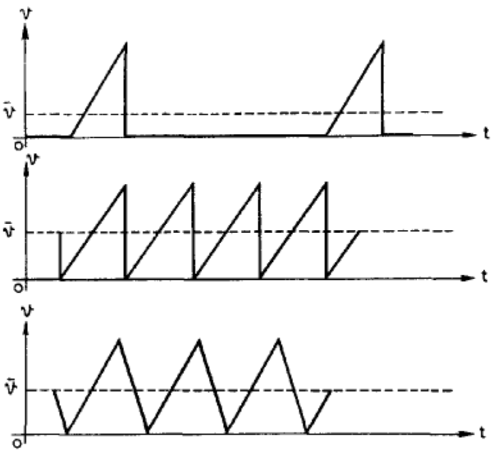


Fig. 44. - Typical waveforms

If a succession of pulses is applied to a series L-R circuit the resulting conditions are comparable with those which obtain in a series C-R circuit. The voltages developed across the coil and resistor are similar to those developed across the resistor and condenser respectively in the C-R arrangement.

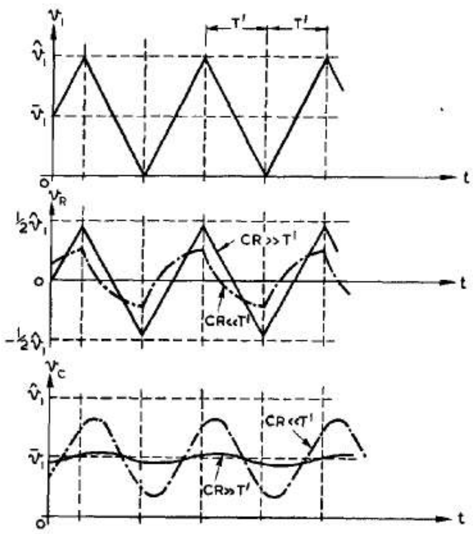


Fig.45.- Response of a C-R circuit to a succession of triangular pulses

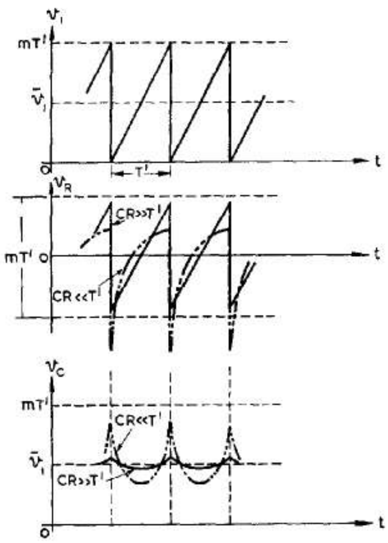


Fig.46.- Response of a C-R circuit to a succession of sawtooth pulses

## 17. DIFFERENTIATING AND INTEGRATING CIRCUITS

With certain limitations, which are discussed below, it is possible to arrange a circuit so that the output provides either a differential or an integral version of the input voltage.

i.e. either $$v_o \propto \frac{dv_i}{dt}$$

or $$v_o \propto \int v_i \, . \, dt$$

A series C-R circuit may be used to meet either of these requirements. For differentiation the time-constant must be short, and the output is taken across the resistor. For integration the time-constant must be long, and the output is taken across the condenser. The conditions governing the choice of time-constant are investigated below.

Considering the circuit of Fig. 47, we know that

$$v_i = v_C + v_R$$

$$i = \frac{dq}{dt}$$

and $$v_C = \frac{q}{C}$$

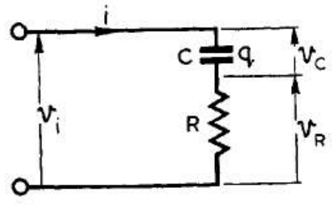


Fig.47.- A differentiating or integrating circuit.

Differentiating Circuit.

Assume that $v_R \ll v_C$

i.e., $v_C \doteq v_i$

then $q/C \doteq v_i$

and by differentiation with respect to time, we obtain

$$\frac{1}{C}\frac{dq}{dt} \doteq \frac{d}{dt}(v_i)$$

Hence $\frac{i}{C} \doteq \frac{d}{dt}(v_i)$

or $\frac{v_R}{CR} \doteq \frac{d}{dt}(v_i)$

i.e. $v_R \doteq CR \frac{d}{dt}(v_i)$

In words, the voltage across the resistor is approximately proportional to the time-rate-of-change of input voltage.

We shall now consider the circumstances under which the assumption

$$v_R \ll v_C$$

is justified.

$$v_R = Ri = \frac{Rdq}{dt} = CR \frac{d}{dt}(v_C).$$

Hence the assumption may be written

$$CR \frac{dv_C}{dt} \ll v_C$$

But, as shown

$$v_C \doteq v_i \quad ;$$

hence

$$CR \frac{dv_i}{dt} \ll v_i \quad ,$$

i.e.

$$CR \ll vi / \frac{dv_i}{dt}$$

This means that the time-constant must be small compared with the ratio of magnitude to time-rate-of-change of the input voltage.

Integrating Circuit

Assume that $v_C \ll v_R$ ,

i.e. $v_R \doteq v_i$ ;

then $Ri \doteq v_i$

and by integration with respect to time we obtain

$$R\int i dt \doteq \int v_i \cdot dt$$

Hence $Rq \doteq \int v_i \cdot dt$

or $CR \; v_C \doteq \int v_i \cdot dt$

i.e. $v_C \doteq \frac{1}{CR}\int v_i \; dt$

In words, the voltage across the condenser is approximately proportional to the time-integral of the input voltage.

It may be shown, by reasoning similar to that used in the case of the differentiating circuit, that the assumption

$$v_C \ll v_R$$

implies that

$$CR \gg \frac{\int v_i \, dt}{v_i}$$

This means that the time-constant must be large compared with the ratio of the integrated value to the magnitude of the input voltage.

As an example of differentiating we may take the circuit and waveform of Fig. 39

Here $\frac{dv_i}{dt} = m$ and is constant, so that $v_i$ increases steadily with time. Hence if t is large enough

$$v_i / \frac{dv_i}{dt} \gg CR$$

and the output $v_R$ is a differentiated version of $v_i$. This is shown in the figure, where, for values of $t \gg CR$, $v_R \propto m$

As an example of integration, consider the waveforms of Fig. 29 For values of $t \ll CR$ the condenser voltage continues to rise with each pulse, so that

$$v_C \propto \int v_i \, dt$$

For larger values of t this condition no longer holds and eventually a steady state is reached. Under these circumstances

$$\int v_i dt \gg CR\, v_i$$

and the condition for integration is not fulfilled.

The terms "differentiating" and "integrating" are often applied to short and long time-constant circuits respectively, even though they are not used in the conditions under which true differentiation or integration occurs. For example, the voltage $v_R$ shown in Fig. 48 is frequently referred to as a differentiated version of the input $v_i$. In fact $\frac{dv_i}{dt}$ is either infinite or zero, as shown. Provided $CR \ll T$, $v_R$ consists of narrow peaks of voltage and the similarity with the true differentiated voltage is obvious.

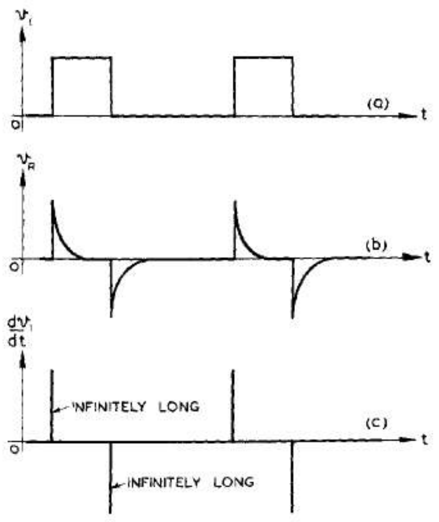


Fig.48. - Comparison between output (b) of a "differentiating" circuit and a true differentiated version (c) of the input (a).


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