CHAPTER V.—SINGLE-PHASE ALTERNATING CURRENTS

INSTANTANEOUS, PEAK AND R.M.S. VALUES

Heating effect of sinusoidal current

1. When an alternating E.M.F. is applied to an electric circuit the immediate result is the production of an alternating current of the same frequency, and in this chapter it is proposed to consider the behaviour of such circuits. The simplest wave-form of an alternating quantity is the sine wave and unless otherwise stated it is always assumed that the current has this wave-form. The value of the current at any instant is called its instantaneous value, while the maximum value attained at any instant during each half-cycle is termed the peak value. The instantaneous value, denoted by i, is related to the peak value $\mathcal{I}$ by the equation

$$i = \mathcal{I} \sin \omega t$$

where $\omega = 2\pi f$.

The instantaneous value of an alternating current or E.M.F. is only measurable by expensive and complicated apparatus, and even if known a single instantaneous value gives no indication of the total effect of the current over a considerable period; the peak value is merely one particular instantaneous value and can only be measured in the same way. Any kind of measuring instrument which indicates the average value of the current, e.g. the moving coil ammeter, must of necessity give a zero reading if connected in an A.C. circuit, for it is obvious from inspection of the wave-form that the average value of the current over any number of complete cycles is zero. The heating effect of an electric current, however, is entirely independent of the direction in which the current is flowing, and the effective value of an alternating current is said to be I amperes if it has the same heating effect as an unvarying current of I amperes. In other words, an alternating current has the same effective value as a given direct current if both produce equal deflection of the pointer in a hot-wire ammeter, so that the latter instrument may be calibrated by comparison with a sub-standard moving coil ammeter in a D.C. circuit, and afterwards used in an A.C. circuit for current measurement. In order to find the value of direct current which is equivalent to a given alternating current $i = \mathcal{I} \sin \omega t$ therefore, it is necessary to find the heating effect of the latter. Now at any instant t, counted from the beginning of a cycle, the power expended, p, (i.e. the rate at which energy is being expended at that particular time) is i^2R joules per second, if R is the total resistance of the circuit and i is the instantaneous value of the current. Hence

$$p = i^2R = R\,\mathcal{I}^2 \sin^2 \omega t$$

The current $i = \mathcal{I} \sin \omega t$, and the instantaneous values of i^2, are plotted (side by side) over a complete period in fig. 1, in which the peak value of the current, $\mathcal{I}$, has been arbitrarily assigned the value 4 amperes. It will be observed that although i passes through both positive and negative values, the curve showing the square of the current has positive values only, which is a

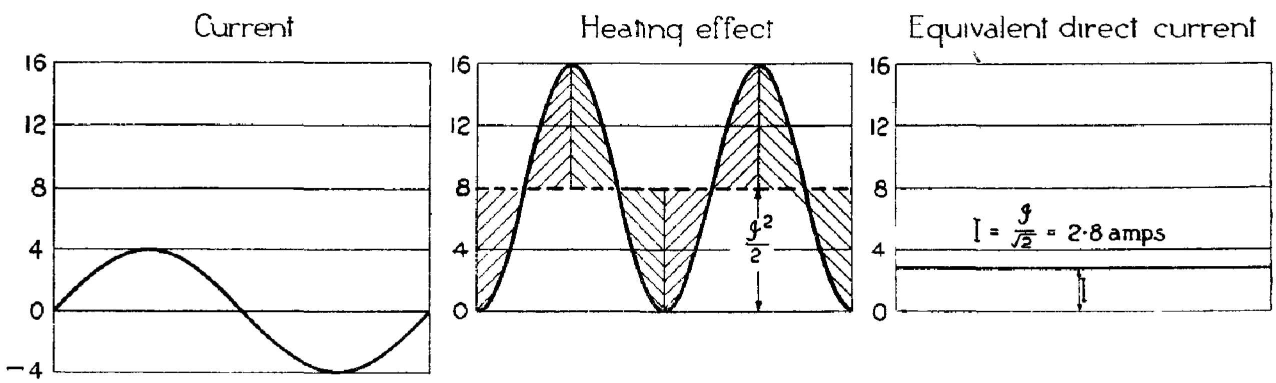


FIG. 1, CHAP. V.—Heating effect of an alternating current.

graphical illustration of the fact that the square of a real quantity is always positive. This curve contains two complete cycles in the time taken for one cycle of i, and its average height is 8 units. This may be verified by tracing the curve on to thin paper, and cutting off the shaded portions above the dotted line. The peaks thus removed will then be found to fit into the hollows left below the line, showing that the area of the whole i^2 curve is $8 \times T$ units, that is $\frac{16}{2} T$ or $\frac{\mathscr{I}^2}{2} T$. The average height of the curve is the area divided by the length of the base, the latter being T units. Hence the average height is $\frac{\mathscr{I}^2}{2}$ units. Another way of arriving at the same conclusion is as follows. The "current-squared" curve is a cosine curve of twice the frequency of the current curve, but its axis is displaced upwards by 8 units, so that it never becomes negative in sign. The curve can therefore be represented by the equation

$$i^2 = \frac{\mathscr{I}^2}{2} - \frac{\mathscr{I}^2}{2} \cos 2\,\omega t$$

the factor $2\,\omega t$ signifying that the frequency of the i^2 curve is twice that of the i curve. $\frac{\mathscr{I}^2}{2}$ is the average value of i^2, the average value of $\frac{\mathscr{I}^2}{2} \cos 2\,\omega t$ over the whole curve being zero. This may be a convenient opportunity to point out that the average value of any function such as $\sin \omega t$, $\sin \theta$, $\cos n\omega t$, etc., for any number of complete periods is always zero.

The heating effect of the current i, in the resistance of R ohms, is therefore given by the equation $P = \frac{\mathscr{I}^2}{2} R$ joules per second, which may be written

$$P = \left(\frac{\mathscr{I}}{\sqrt{2}}\right)^2 R \text{ watts.}$$

Let us now compare two currents having equal heating effect, a direct current of unvarying value I, and an alternating current of peak value $\mathscr{I}$, which are assumed to flow through a resistance of R ohms. The steady current has a heating effect of I^2R watts, and the alternating current a heating effect of $\left(\frac{\mathscr{I}}{\sqrt{2}}\right)^2 R$ watts. The heating effect of the two currents will be equal if

$$I^2 = \left(\frac{\mathscr{I}}{\sqrt{2}}\right)^2, \text{ or } I = \frac{\mathscr{I}}{\sqrt{2}}$$

The expression $\frac{\mathscr{I}}{\sqrt{2}} = \frac{\mathscr{I}}{1 \cdot 414}$ or $\cdot 707\ \mathscr{I}$ is called the root-mean-square or R.M.S. value of the current whose peak value is $\mathscr{I}$. Whenever an alternating current is said to have a value I amperes, without further qualification, the R.M.S. value is implied. Also, since the heating effect of a direct current is equal to $\frac{V^2}{R}$ joules per second, the R.M.S. value of an alternating voltage of peak value $\mathscr{V}$ is $\frac{\mathscr{V}}{\sqrt{2}}$. When used in this way the factor $\sqrt{2}$ is called the peak factor. It will be observed that three conventions of notation have been introduced in the preceding discussion, the instantaneous value of an alternating quantity being denoted by a small italic letter, the peak value by a cursive or script capital, and the R.M.S. value by an italic capital. This notation will be followed throughout this chapter.

Wave-Form.

2. Up to the present we have considered only those alternating quantities which obey the simple sine law, but the shape of the graph obtained by plotting the instantaneous value or displacement of the quantity at various intervals of time may take any one of an infinite variety of forms, and this displacement-time curve may be referred to as the wave-form of the quantity.

Provided that the wave-form repeats itself at regular intervals, it can be proved that it is built up by the addition of a number of simple sine waves of various frequencies. The frequency of each component sine wave is an integral multiple of some fundamental frequency (which may or may not be present), and is said to be in harmonic relation with the latter, while the higher frequencies are called the harmonics of the fundamental. If the average value of the quantity over any number of complete periods is not zero, a constant displacement must also exist; for instance, if the quantity under consideration is an alternating current, the wave-form being complex and repetitive, it may be represented by an equation of the form

$$i = I_0 + \mathcal{I}_1 \sin(\omega t + \varphi_1) + \mathcal{I}_2 \sin(2\,\omega t + \varphi_2) + \mathcal{I}_3 \sin(3\,\omega t + \varphi_3) + \ldots .$$

I_0 is the average value of the current over any number of complete periods and may be regarded as a direct current superimposed upon the alternating components. The fundamental frequency is $\frac{\omega}{2\pi}$ and this is also referred to as the first harmonic. The second harmonic has a frequency of twice the fundamental or $\frac{2\omega}{2\pi}$, while the third harmonic frequency is $\frac{3\omega}{2\pi}$ and so on. The angles represented by the symbols $\varphi_1, \varphi_2, \varphi_3 \ldots\ldots$ are inserted to signify that it is not necessary for all the components to pass through zero displacement at the instant arbitrarily assumed to be zero time.

3. Frequencies which are even multiples of the fundamental frequency f, e.g. $2f$, $4f$, etc. are called the even harmonics, and those which are odd multiples, $3f$, $5f$, etc. are called the odd harmonics. An E.M.F. generated by rotating machinery is free from even harmonics, but may

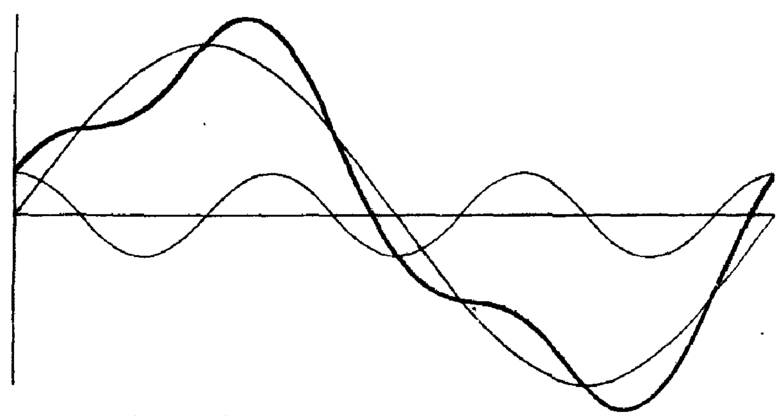


FIG. 2, CHAP. V.—Wave with third harmonic.

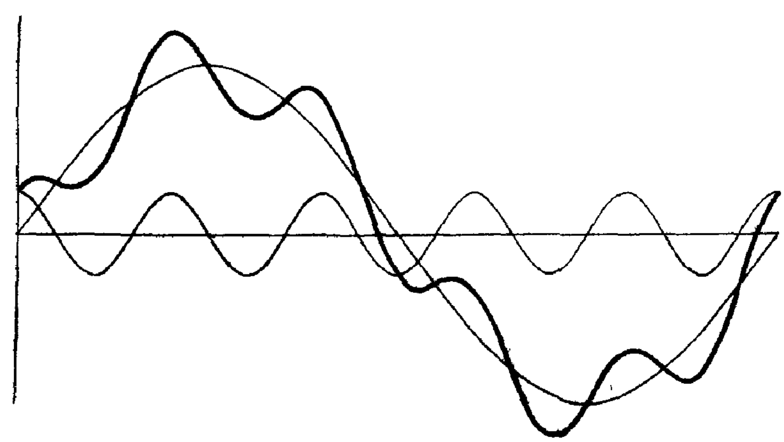


FIG. 3, CHAP. V.—Wave with fifth harmonic.

contain odd harmonics. This entails that the positive and negative half-cycles have exactly the same wave-form, which must be true if the E.M.F. is produced by rotation of a conductor in an unvarying magnetic field. The odd harmonics are produced by uneven flux distribution, and do not occur if the field is uniform as it was assumed to be when considering the production of an alternating E.M.F. in the preceding chapter. Typical wave-forms containing odd harmonics only are portrayed in figs. 2 and 3. Even harmonics are frequently found in radio circuits, a particular instance being the alternating current flowing in the anode circuit of a valve transmitter.

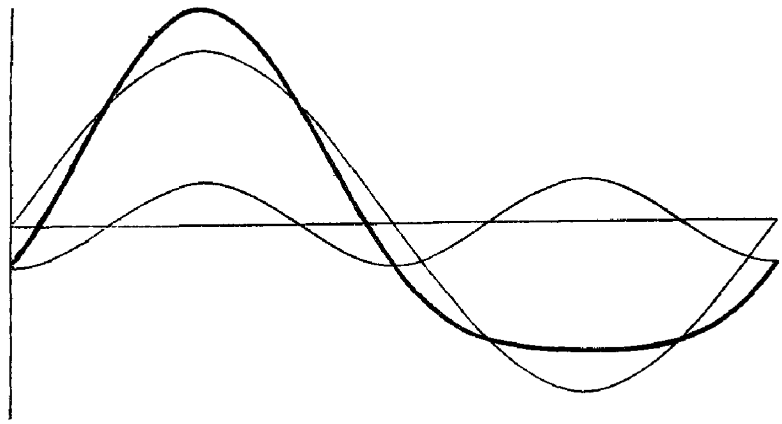


FIG. 4, CHAP. V.—Wave with second harmonic.

The existence of an even harmonic in a given wave-form may be instantly detected because its presence causes the negative half-cycles to be of a shape differing from the positive half cycles. The point is illustrated in fig. 4 which shows the result of adding to a sine wave a second harmonic which bears a certain phase relationship to the fundamental. This wave-form, with the addition of further harmonics of small peak value, is often met with. The wave-form resulting from the addition of the two components mentioned may be represented by the equation

$$i = \mathscr{I} \sin \omega t - \frac{\mathscr{I}}{2} \sin\left(2\omega t + \frac{\pi}{2}\right) .$$

Effective value of current of complex wave-form

4. In order to find the heating effect of a current of non-sinusoidal but recurring wave-form we may utilise the result already obtained for the simple sine wave. If the current flows through a resistance of R ohms, the component current of fundamental frequency will cause energy to be expended at a rate of $\frac{\mathscr{I}_1^2 R}{2}$ joules per second, while the second harmonic component will cause an expenditure of energy at a rate of $\frac{\mathscr{I}_2^2 R}{2}$ joules per second and so on. The total energy expended will therefore be given by the expression

$$P = I_0^2 R + \frac{\mathscr{I}_1^2}{2} R + \frac{\mathscr{I}_2^2}{2} R + \frac{\mathscr{I}_3^2}{2} R \ldots\ldots$$
$$= R\left\{I_0^2 + \left(\frac{\mathscr{I}_1}{\sqrt{2}}\right)^2 + \left(\frac{\mathscr{I}_2}{\sqrt{2}}\right)^2 + \left(\frac{\mathscr{I}_3}{\sqrt{2}}\right)^2 \ldots\ldots\right\}$$

But this is equal to $I^2 R$, if I is the effective value of the current. Hence

$$I = \sqrt{\left\{I_0^2 + \left(\frac{\mathscr{I}_1}{\sqrt{2}}\right)^2 + \left(\frac{\mathscr{I}_2}{\sqrt{2}}\right)^2 + \left(\frac{\mathscr{I}_3}{\sqrt{2}}\right)^2 \ldots\ldots\right\}}$$

or if I_1, I_2, I_3 etc. are the R.M.S. values of the currents of peak value $\mathscr{I}_1$, $\mathscr{I}_2$, $\mathscr{I}_3$ etc.

$$I = \sqrt{\left\{I_0^2 + I_1^2 + I_2^2 + I_3^2 \ldots\ldots\right\}}$$

The peak factor for the complex wave is defined in the same way as before, being the ratio of the peak value to the R.M.S. value. The mean value of a single half-cycle of an alternating current is sometimes required. Consider first the simple sine wave ; we may obtain an approach to the average value of $\sin\theta°$, from $\theta = 0$ to $\theta = 90°$, by means of a table of sines, adding up the whole series and dividing by the number of values given in the table. This is a laborious process, but if actually performed it will be found that the average value of $\sin\theta$ from 0 to 180° is very near to ·637. Actually the calculation can be performed with much less labour, although the mathematical ideas involved are more complicated. The result of such a calculation gives the average value of $\sin\theta$ during one half-cycle as $\frac{2}{\pi}$ which is ·6366. Hence the average value of a sine wave over one half-cycle is $\frac{2}{\pi}$ times the peak value. The ratio $\frac{\text{R.M.S. value}}{\text{mean value}}$ is called the form factor, and is equal to 1·11 for a sinusoidal wave.

For any wave-form whatever, the average value per half-cycle may be obtained by drawing the wave to some convenient scale, calculating its area using the mid-ordinate method or Simpson's rule, and dividing the area by the length of the base line which represents one half period. An approximation to the average and R.M.S. values may also be obtained from the instantaneous values at a number of equal intervals during the cycle, as in example (2) below.

Examples.—(1) An alternating current is represented by the equation

$$i = 250 \sin \omega t + 125 \sin 3\,\omega t + 50 \sin 5\,\omega t.$$

Find its R.M.S. value.

Here $I_0 = 0, \mathcal{I}_1 = 250, \mathcal{I}_2 = 0, \mathcal{I}_3 = 125, \mathcal{I}_4 = 0, \mathcal{I}_5 = 50.$

$$I = \sqrt{\frac{250^2}{2} + \frac{125^2}{2} + \frac{50^2}{2}}$$
$$= \sqrt{31250 + 7812{\cdot}5 + 1250}$$
$$= \sqrt{40312{\cdot}5}$$
$$= 201 \text{ amperes (nearly).}$$

(2) An alternating voltage is found to pass through the instantaneous values given in the following table.

Angle ..	0°	15°	30°	45°	60°	75°	90°
Voltage ..	0	25	50	88	130	175	225
Angle ..	105°	120°	135°	150°	165°	180°	195° etc.
Voltage ..	270	230	165	80	30	0	—25 etc.

the negative half-cycle being of the same shape as the positive.

Find an approximation to the R.M.S. and average values, and the peak and form factors.

The average value of one half-cycle is

$$\frac{1}{12}\left[0 + 25 + 50 + 88 + 130 + 175 + 225 + 270 + 230 + 165 + 80 + 30\right]$$
$$= \frac{1468}{12} = 122 \text{ volts.}$$

The ratio $\frac{\text{mean value}}{\text{peak value}}$ is $\frac{1221\frac{1}{3}}{270}$ = ·453 which may be compared with the value ·637 for a sinusoidal wave. The R.M.S. value is the square root of the mean of all the squares of the above ordinates, i.e.

$$V^2 = \frac{1}{12}\left[\begin{array}{l}0 + 625 + 2500 + 7744 + 16900 + 30625 + 50625 + 72900 + 52900 \\ \quad + 27225 + 6400 + 900\end{array}\right]$$
$$= \frac{270344}{12} = 22529$$
$$V = \sqrt{22529}$$
$$= 150{\cdot}9 \text{ volts.}$$

The peak factor, $\frac{\text{peak value}}{\text{R.M.S. value}} = \frac{270}{150 \cdot 9} = 1 \cdot 79$ which again should be compared with the value $1 \cdot 414$ for a sinusoidal wave.

The form factor, $\frac{\text{R.M.S. value}}{\text{mean value}} = \frac{150 \cdot 9}{122 \cdot 3} = 1 \cdot 23$. If the wave-form is plotted from the data it will be found to be more peaked than a sine wave. Such a wave has high values for its peak and form factors while the converse is true of flat-topped waves.

MEASUREMENT OF CURRENT, VOLTAGE AND FREQUENCY

Ammeters and voltmeters

5. From the foregoing, it will be appreciated that any type of ammeter or voltmeter in which the deflection is proportional to the average value of the current or voltage is unsuitable for use in A.C. circuits. Hot-wire ammeters, thermo-ammeters and moving iron ammeters may be used for measurement of alternating current, and hot-wire and electrostatic voltmeters for the measurement of alternating P.D. In general it may be stated that instruments which depend for their action upon any form of permanent magnet are unsuitable for use in A.C. circuits. Instruments depending upon magnetisation by the current, for example moving iron instruments, can be used for A.C. measurement, but special care is necessary in the design, in order to reduce the effects of eddy currents. This necessitates the sub-division of any metallic parts in proximity with the current-carrying conductors, and may be explained with reference to the repulsion type of ammeter. If the coil is wound upon a metal former, it is necessary that the former should be cut completely in a radial direction, so that eddy currents cannot circulate completely round the coil former. The iron portions may also be laminated for the same reason. Theoretically, if such an instrument is calibrated with steady current its scale should read R.M.S. value when connected in an A.C. circuit, but in practice this is not quite true, partly on account of the varying permeability of the iron with varying magnetising current, and partly owing to the inductance of the instrument which has no effect when the instrument is used to measure direct current. It is therefore desirable that such instruments should be calibrated by alternating current of the wave-form and frequency with which they are to be employed.

6. Hot-wire instruments have the great advantage that the deflection is independent of wave-form and frequency, provided that the resistance of the instrument is the same at all frequencies. It may be assumed that a hot-wire instrument calibrated at say 250 cycles per second, may be used without fear of serious inaccuracy on frequencies between 25 and 500 cycles per second, but may be in serious error at higher frequencies.

Dynamometer instruments are moving coil instruments of special design. Instead of being established by a permanent magnet as in the instruments commonly termed "moving coil type" the magnetic field is supplied by a fixed winding carrying an alternating current, this winding being connected in series with that of the moving coil. These instruments are expensive, and offer no advantages over hot-wire instruments as ammeters or voltmeters, but by a simple modification the principle is employed in the construction of one form of watt-meter, which is described later.

Frequency meters

7. Two forms of frequency meter are found in low (or commercial) frequency circuits, namely the tuned reed pattern and the induction type. In the tuned reed pattern (fig. 5) a number of steel strips are so adjusted that each vibrates at a particular frequency. A laminated soft iron core carries a magnetising winding consisting of a great many turns of fine insulated wire, the winding being connected across the supply mains in the same way as a voltmeter, and the steel strips or reeds are arranged in such a manner that they are acted upon by this electromagnet. Any reed which is adjusted to the frequency of the supply at any particular

instant is caused to vibrate with appreciable amplitude, although reeds of different frequency are scarcely affected. To facilitate observation, each reed carries at its free end a small rectangular metal flag, and the frequency is obtained by noting on the adjacent scale the frequency corresponding to the reed having maximum amplitude of vibration. The induction type (fig. 6)

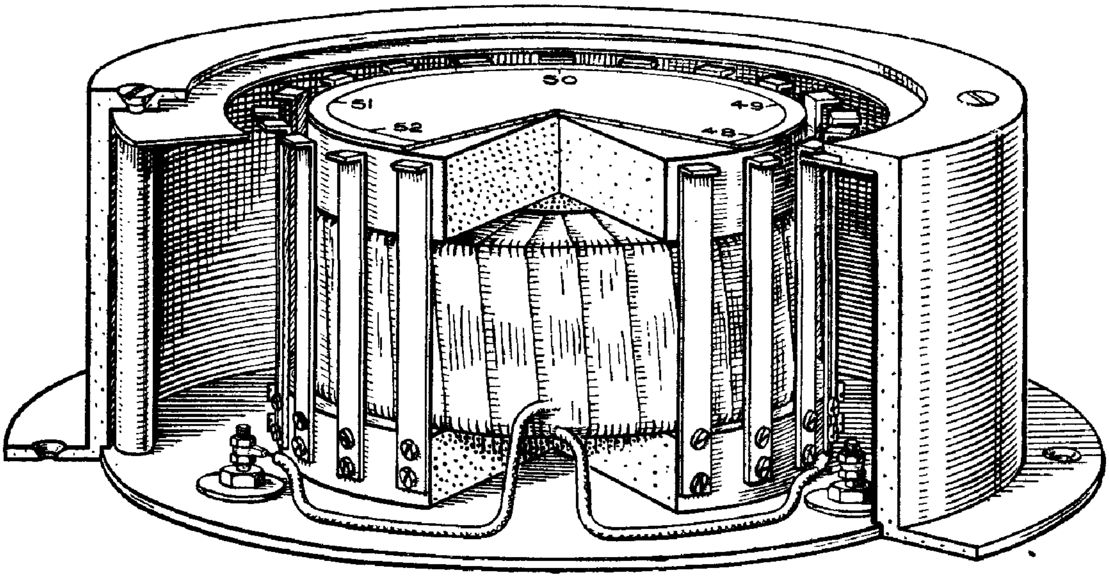


FIG. 5, CHAP. V.—Frequency meter, tuned reed pattern.

consists of two coils which are mounted at right-angles and on a common axis. One of the windings has connected in series a large non-inductive resistance, while the other has a low resistance coil of large inductance in series. The two elements thus formed are each connected across the supply mains. These two coils co-operate in the establishment of a magnetic field in the space enclosed by the coils, and the direction of this field depends upon the ratio of

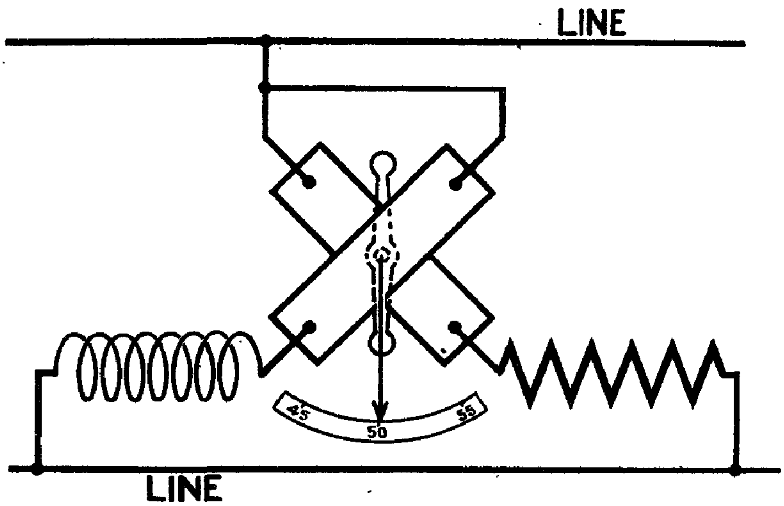


FIG. 6, CHAP. V.—Frequency meter, induction pattern.

current in the two windings. As will be seen after a study of the following sections, a change of frequency will make little or no difference to the current in the predominantly resistive branch, but the current in the winding which is predominatingly inductive will vary inversely as the frequency. Hence the direction of the resulting field varies with frequency, and this direction will be taken up by a soft iron needle which is mounted upon a spindle on the common axis of the coils. A pointer attached to the spindle moves over a scale which is graduated in cycles per second.

The term frequency meter is also sometimes employed to denote an instrument which is used to measure very high frequencies (of the order of 10^4 cycles and above). Such instruments are described in a later chapter.

REPRESENTATION OF ALTERNATING QUANTITIES BY VECTORS

Vector quantities

8. A vector quantity is one which has direction as well as magnitude, for example, the magnetic field strength H at any point in the field is a vector quantity. Those quantities which have magnitude only, e.g. work, are called scalar quantities. A vector quantity may be represented in magnitude and direction by a straight line, and such a line is often referred to as "the vector" representing the quantity. When used in connection with alternating quantities, vectors are used in a somewhat special manner. The line is supposed to be fixed at one end and to rotate at the frequency of the alternation. For example, consider a voltage $v = \mathscr{V} \sin \omega t$. This voltage may be represented by a line of length $\mathscr{V}$ units, rotating in an anticlockwise direction with reference to an arbitrary datum line or reference vector, which is usually drawn horizontally to the right of the centre of rotation as shown in fig. 7. After any time t seconds, the instantaneous value of the voltage, v, is shown by the height of the vertical projection of the end of the vector. In fig. 7 the vector has rotated through an angle of ωt

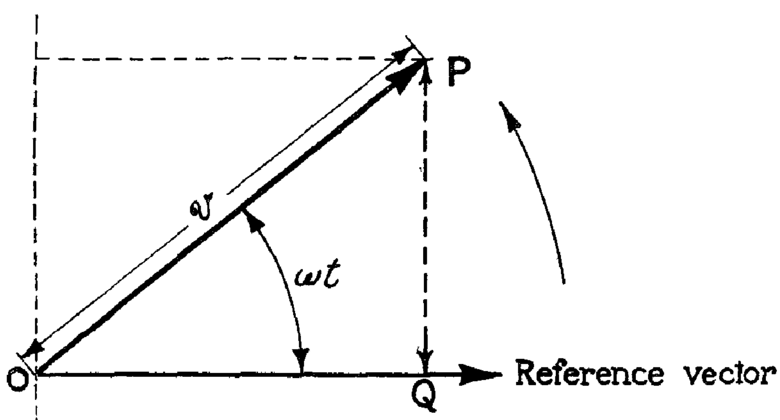


FIG. 7, CHAP. V.—Rotating vector.

radians, and the instantaneous voltage is shown to the same scale as the vector $\mathscr{V}$ by the line PQ which is equivalent to the projection of its length upon the vertical axis. In the study of alternating currents the advantage of vector representation is its ability to depict directly the conception of phase difference. When two such quantities of the same frequency pass through corresponding points in the wave-form at the same instant, they are said to be in phase with each other, while if they pass through corresponding points at different instants, there is said to be a phase difference between them, and one is said to be leading or lagging on the other. In order to show this a simple example may be taken, thus let $\mathscr{E}$ represent the peak value of an alternating E.M.F., and $\mathscr{I}$ the corresponding current, then if $\mathscr{E}$ and $\mathscr{I}$ are in phase they may be considered as vectors which are coincident in direction and rotate at equal speed. $\mathscr{E}$ and $\mathscr{I}$ are therefore represented as in fig. 8a. Actually $\mathscr{I}$ lies upon $\mathscr{E}$, but for clearness the two vectors are drawn side by side. In practice $\mathscr{E}$ may either be in phase with, or lead or lag upon the resulting current; in fig. 8b $\mathcal{I}$ is shown as leading, and in fig. 8c as lagging, with reference to the E.M.F. By convention the direction of rotation is always anticlockwise, and is shown on the vector diagram by a curved arrow.

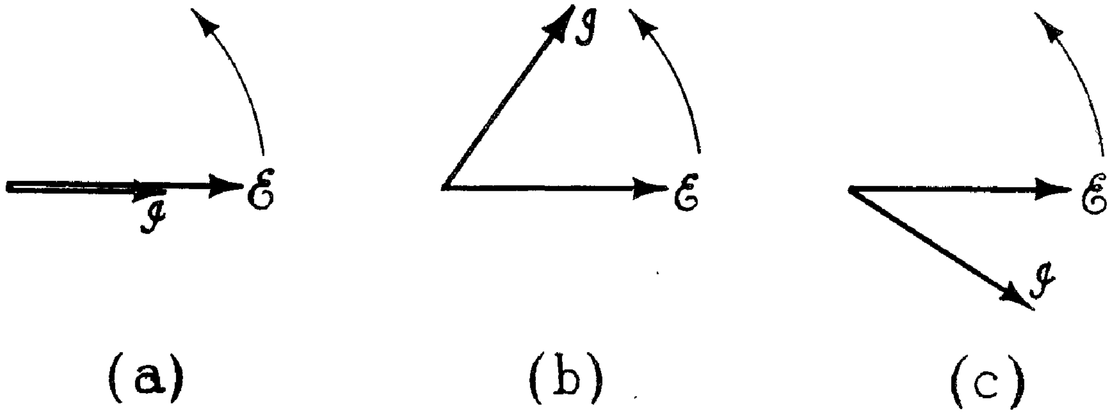


FIG. 8, CHAP. V.—Vector representation of phase difference.

Although the principle of vector representation is based upon the peak value, vector diagrams may be drawn to a scale of R.M.S. values, but it must then be remembered that the projection of the vector upon the vertical does not give the instantaneous value; although the phase difference between various quantities of the same frequency is still shown.

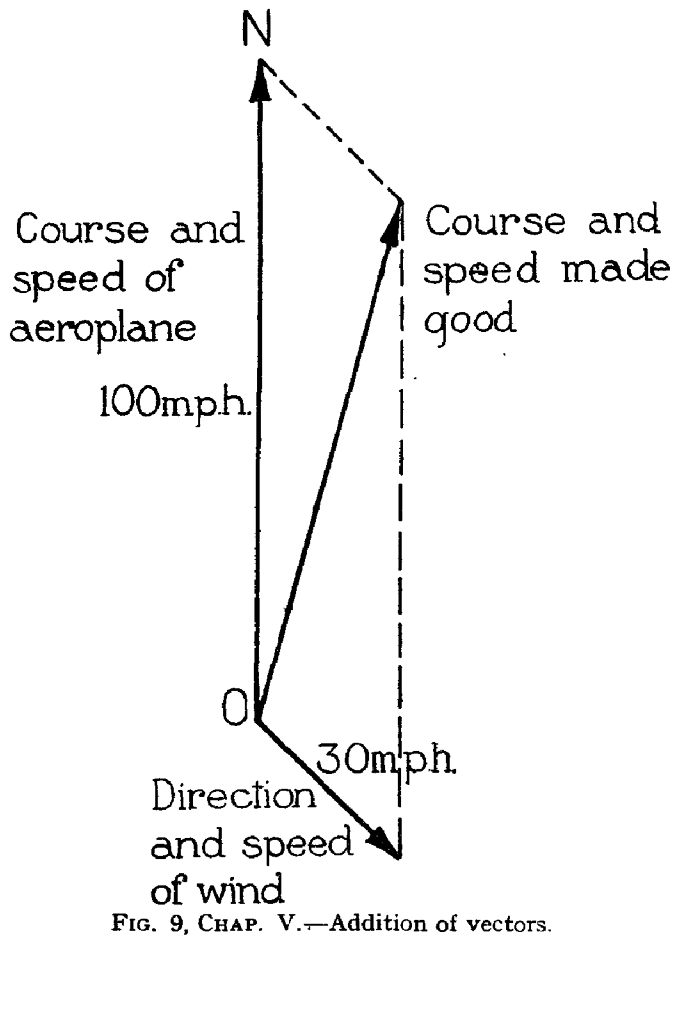


FIG. 9, CHAP. V.—Addition of vectors.

9. Like scalar or arithmetical quantities vectors may be added and subtracted. For instance, an aeroplane may have an actual speed of 100 miles per hour in a direction true North, but a wind from the North-west of 30 miles an hour may also act upon the machine. The velocity of the aeroplane with reference to the ground will then be the vector sum of the two velocities, and may be found by drawing the two vectors to scale, as shown in fig. 9. The two vectors form two sides of a parallelogram, and the vector sum can be found by completing the parallelogram (as shown by dotted lines) and then drawing the diagonal through the origin. The latter then represents the sum of the two vectors in magnitude and direction. Subtraction of vectors is carried out

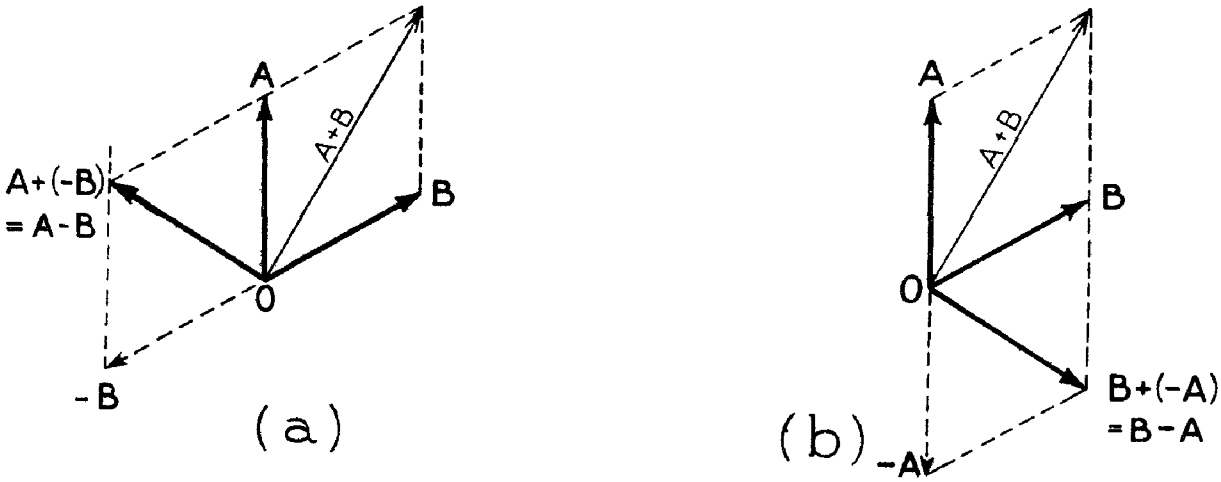


FIG. 10, CHAP. V.—Subtraction of vectors.

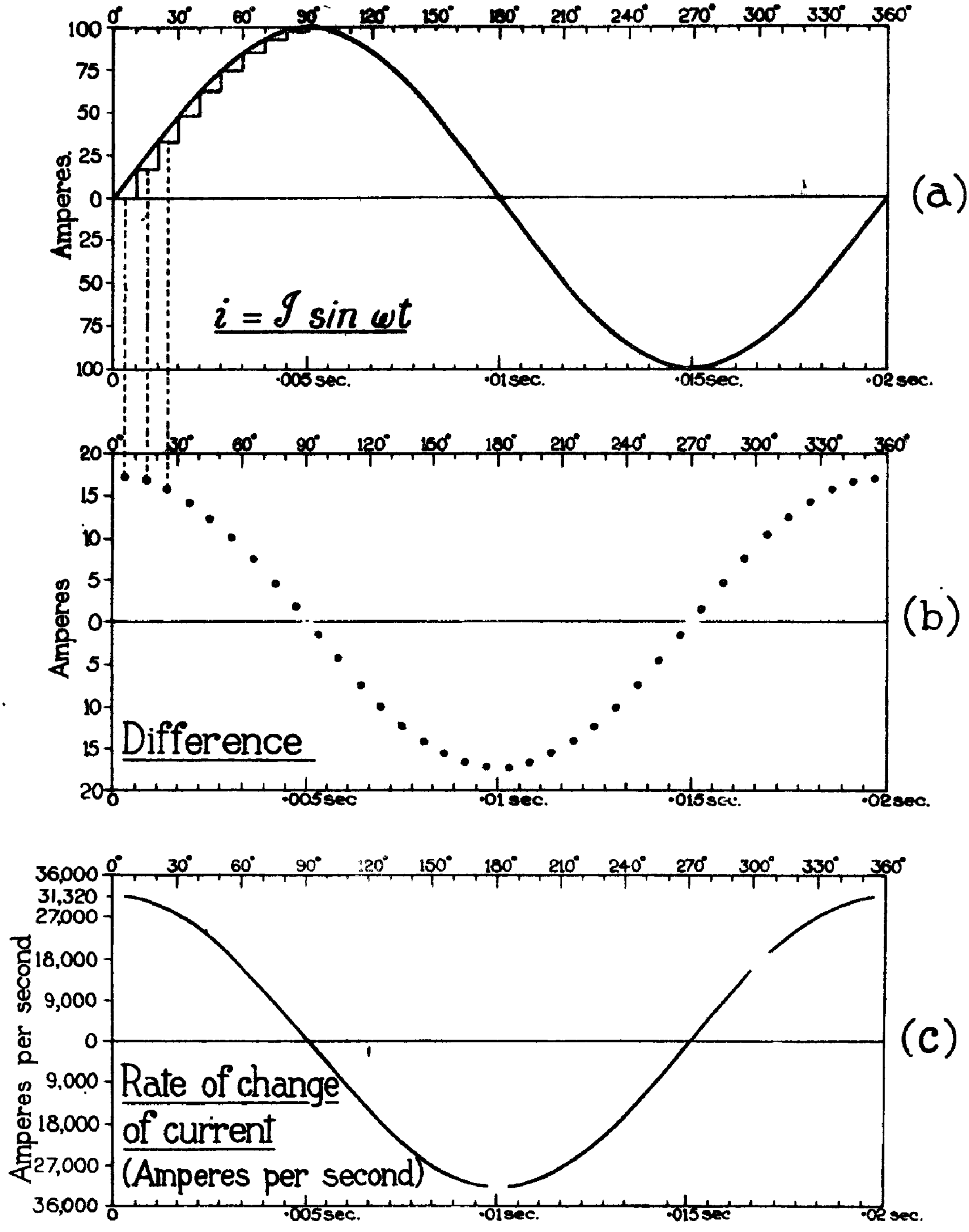
by the following method. Suppose that A and B are two vectors, and the vector difference A–B is required. This may be written A + (–B), and this expression gives the clue to the graphical solution, which is as follows. Draw the vectors A and B, then draw a vector equal in magnitude to B, but contrary in direction. This vector is then equal to –B. Perform the addition of the vectors A and –B by the parallelogram rule given above. The diagonal of this parallelogram passing through the origin is equal to the vector A–B, and it must be clearly understood that the vector A–B is not equal to the vector B–A, because its direction is different, as will be understood by reference to fig. 10.

The rate of change of a sinusoidal quantity

10. In earlier chapters, several illustrations of Faraday's law have been met, and it will be remembered that in its mathematical form the law is

$$e = L\frac{di}{dt}$$

Hitherto the symbol $\frac{di}{dt}$ has been considered merely as an abbreviation of the phrase " the rate of change of current with respect to time ". The unit of $\frac{di}{dt}$ is the ampere per second, and its numerical value must be known before the self-induced E.M.F. can be calculated. Since an alternating current is constantly varying in magnitude, it becomes necessary to evaluate this quantity for a sinusoidal wave, and in order to exhibit the principle of the method a definite example may be taken. Fig. 11 (*a*) shews a sinusoidal current having a peak value of 100 amperes, and it is required to find the rate of change at every instant during the cycle. The period is ·02 second, and this has been divided into 36 equal time intervals, each of $\frac{1}{1800}$ second ; since one cycle is equal to 360° each interval corresponds to an angle of 10°. The instantaneous value



CRAPHICAL DERIVATION OF APPROXIMATE VALUE

OF $\frac{di}{dt}$ WHEN $i = \mathscr{I} \sin \omega t$

FIG. 11
CHAP V

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of the current at each of these intervals in the first quarter of a cycle is given in the table below, and these values will be repeated, in the reverse order, during the second quarter of the cycle. Between 180° and 360° the current values will be a repetition of those in the first half-cycle but with negative sign.

Angle ..	0°	10°	20°	30°	40°	50°	60°	70°	80°	90°
i	0	17·4	34·2	50	64·3	76·6	86·6	94	98·5	100
Difference ..	17·4	16·8	15·8	14·3	12·3	10	7·4	4·5	1·5	

In the third line of the table the change of current during each interval is given. During the first interval, the current increases from zero to 17·4 amperes, during the second from 17·4 to 34·2, an increase of 16·8 amperes, and so on. These successive increments of current are plotted in fig. 11b.

The average rate of change of current during each interval is found by dividing the change of current by the duration of the interval. For example during the first 10° the average rate of change is $17{\cdot}4 \div \frac{1}{1800}$ or 31320 amperes per second. During the next 10° the average rate of change is $16{\cdot}8 \div \frac{1}{1800}$ or 30240 amperes per second. On repeating this process for the whole of the first quarter of a cycle it will be seen that the rate of change of current is greatest where the current itself is small, but as the current increases the rate of change decreases. At the end of the first quarter of a cycle, the current reaches its peak value and is momentarily neither increasing nor decreasing. Its rate of change is therefore zero, and is so plotted in fig. 11b.

During the second quarter of a cycle the current is decreasing in value and its rate of change must be regarded as negative in sign, while during the third quarter the current is increasing in value but is of negative sign, and its rate of change must be regarded as negative. During the fourth quarter the current is decreasing in value and is still of negative sign so that the rate of change is again positive. The average rate of change of current has been plotted in fig. 11c and a smooth curve drawn through the points. This curve is incomplete because no values have been obtained for the rate of change at the instants $t = 0$, $t = \cdot 01$ second and $t = \cdot 02$ second. It can be seen nevertheless that it is very nearly a sine curve moved through a quarter of a cycle, that is, a cosine curve. If the period were divided into a larger number of intervals before carrying out the above process, the approximation would be closer still. It may now be stated that the instantaneous value of the rate of change of a sinusoidal current, $i = \mathscr{I} \sin \omega t$ is a co-sinusoidal quantity and may be represented by the equation

$$\frac{di}{dt} = K \cos \omega t$$

where K is the maximum rate of change. The maximum rate of change occurs when the current itself is passing through the value zero, but its exact value cannot be obtained by the method used above. It can however be deduced as follows. Referring to fig. 12, let θ be the magnitude of the angle ROP in radians, and the length of the radius OR be r. Since the radian measure of an angle is given by the ratio $\frac{\text{arc}}{\text{radius}}$, $\theta = \frac{arc\ \text{RP}}{r}$, while $\sin \theta = \frac{\text{QP}}{r}$. It is obvious that the arc RP is of greater length than the perpendicular QP, and therefore that $\frac{\sin \theta}{\theta}$ is less than unity. Now allow the angle θ to decrease to θ' by rotating OP into the position OP′. The magnitude of θ' is given by the ratio $\frac{arc\ \text{RP}'}{r}$ while $\sin \theta' = \frac{\text{Q}'\text{P}'}{r}$ and $\frac{\sin \theta'}{\theta'}$ although smaller than unity has approached nearer to that value. In the lower diagram of fig. 12 the angle θ has been made still smaller; the arc RP and perpendicular QP are now very nearly of the same length, and it is deduced that if the angle decreases without limit, or approaches the value zero, the ratio $\frac{\sin\theta}{\theta}$ approaches the value unity. This conception is applied in finding the peak value of the rate of change of current by taking a small interval of time, δt, measured from the time $t = 0$. As $i = \mathcal{I} \sin \omega t$, at the end of the interval δt the current will be equal to $\mathcal{I} \sin \omega\, \delta t$ and the average rate of change $\frac{\delta i}{\delta t}$ during the interval will be $\frac{\mathcal{I} \sin \omega\, \delta t}{\delta t}$. As the interval δt is made smaller and smaller so the expression $\mathcal{I} \sin \omega\, \delta t$ becomes more nearly equal to $\mathcal{I}\ \omega\, \delta t$ and

$$\frac{\delta i}{\delta t} = \frac{\mathcal{I}\ \omega\, \delta t}{\delta t} = \omega\, \mathcal{I}$$

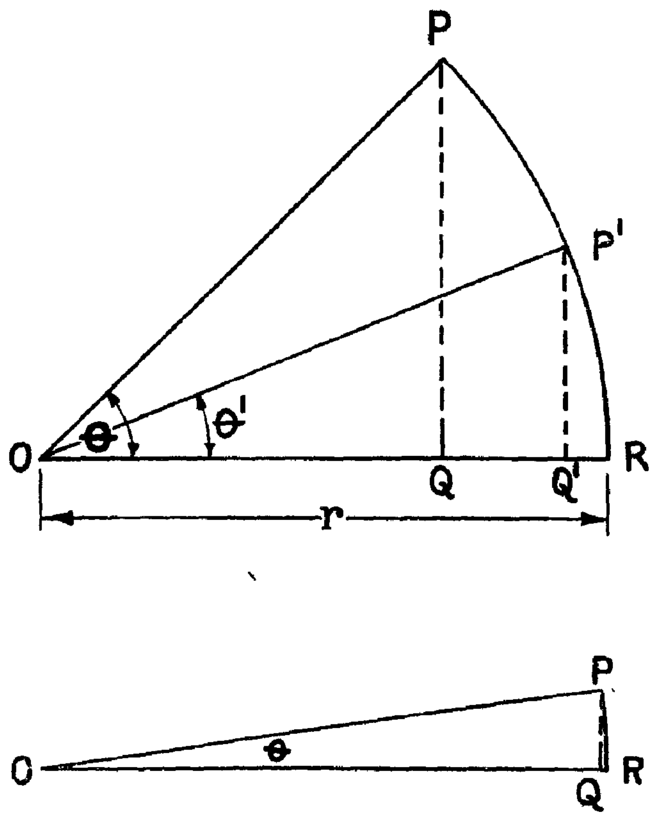


FIG. 12, CHAP. V.—Relation between *arc θ sin θ*.

This is the maximum value of the rate of change of current. Hence, if

$$i = \mathcal{I} \sin \omega t$$

$$\frac{di}{dt} = \omega\, \mathcal{I} \cos \omega t$$

In the above example, $f = \frac{1}{T} = 50$ cycles per second and the maximum rate of change of current is $2\pi \times 50 \times 100$ or 31416 amperes per second.

11. It is often necessary in alternating current work to find the rate of change of a quantity. Let us first consider a simple instance of varying velocity taking a motor car starting from rest as an example. On letting in the clutch, the car starts to move, with increasing velocity. Now the velocity is the rate at which its position is changing, while the

rate at which its velocity is increasing is the acceleration. Thus a certain car was found to increase its speed from 10 to 31 miles per hour in seven seconds, and its acceleration is $\frac{31-10}{7}\ \frac{\text{miles per hour}}{\text{sec.}}$ or since 1 mile $= 5280$ feet, the acceleration is $\frac{21 \times 5280}{3600 \times 7}$, or 4·4 ft. per second per second. When the car has travelled x feet its velocity v is $\frac{dx}{dt}$, while its acceleration is the rate at which its velocity is changing, or $\frac{dv}{dt}$. Now as $v = \frac{dx}{dt}$, the acceleration is $\frac{d}{dt}\left(\frac{dx}{dt}\right)$ and as a matter of convenience this is written $\frac{d^2x}{dt^2}$. It must be fully realised that the indices are not " powers " to which the quantities have to be raised, e.g. d^2 is not " d-squared " ; the symbol $\frac{d^2x}{dt^2}$ must be regarded merely as a kind of shorthand sign for " the acceleration of a body in the direction x ".

In the case of a sinusoidal quantity such as $i = \mathcal{I} \sin \omega t$ the rate of change of which is $\frac{di}{dt} = \omega \mathcal{I} \cos \omega t$, to find the rate of change of the latter it is necessary to repeat the procedure used in finding $\frac{di}{dt}$ but operating upon the cosine curve instead of the sine curve. The detailed process will not be repeated, but the results can be stated thus :—

$$\text{If} \qquad i = \mathcal{I} \sin \omega t$$

$$\frac{di}{dt} = \omega \mathcal{I} \cos \omega t$$

$$\frac{d^2i}{dt^2} = -\omega^2 \mathcal{I} \sin \omega t = -\omega^2 i$$

ALTERNATING CURRENT CIRCUITS

A.C. resistance

12. When energy is supplied to any electrical circuit, some portion may be expended in doing useful work, but a portion is always expended in producing heat. If a circuit has a resistance R and the R.M.S. current flowing through it is I amperes, energy is converted into heat at a rate of $I^2 R$ joules per second. In direct current circuits, the resistance of any conductor can be calculated from the relation $R = \frac{l\varrho}{A}$. In alternating current circuits, however, the heating effect is not confined to the conductor, for any and every other conductor situated within the magnetic field set up by an alternating current is the seat of currents due to the induced E.M.F. (Faraday's law) and consequently energy is converted into heat in these conductors. If any ferro-magnetic material is situated in this field, the continual reversal of the direction of magnetisation necessitates a further expenditure of energy, giving rise to what is called the hysteresis loss. The rate at which energy is dissipated by induced currents and by hysteresis depends upon the frequency of the A.C. supply. Energy may also be dissipated in the form of radiation. The consideration of this phenomenon is deferred until a later chapter, but the position may be summarised in the statement that for alternating current practice a new definition of resistance must be introduced, namely: The total effective resistance of an A.C. circuit is that quantity which, when multiplied by the square of the R.M.S. current, is equal to the rate at which energy is dissipated. If P is the rate of energy dissipation in joules per second or watts,

$$P = I^2 R$$

$$\text{and} \qquad R = \frac{P}{I^2}$$

The total effective resistance includes that due to all causes of power dissipation. As copper is almost universally employed for electrical conductors the power loss due to the conductor itself is often referred to as the copper loss. This in turn may be divided into three components, (i) the D.C. resistance, given by the formula $R = \frac{l\varrho}{A}$ (ii) the resistance due to a phenomenon called skin effect, which may be described as a tendency for the current to flow on the surface of the conductor instead of being uniformly distributed over its cross-section, and (iii) that due to what is termed proximity effect, which causes the current to concentrate in those portions of the conductor which are most remote from other conductors. Both skin and proximity effects are caused by the induction of eddy currents. Fig. 13 shows a portion of a conductor carrying an alternating current which is represented by solid lines parallel to the axis of the wire. The current gives rise to an alternating magnetic field, both inside and outside the conductor, and therefore to induced E.M.F.'s which cause eddy currents to circulate in paths somewhat as shown by the dotted lines, in opposition to the main current in the centre of the wire but in the same direction at the surface. The effect of this non-uniform current distribution is to cause an increase in the effective resistance of the conductor. To understand

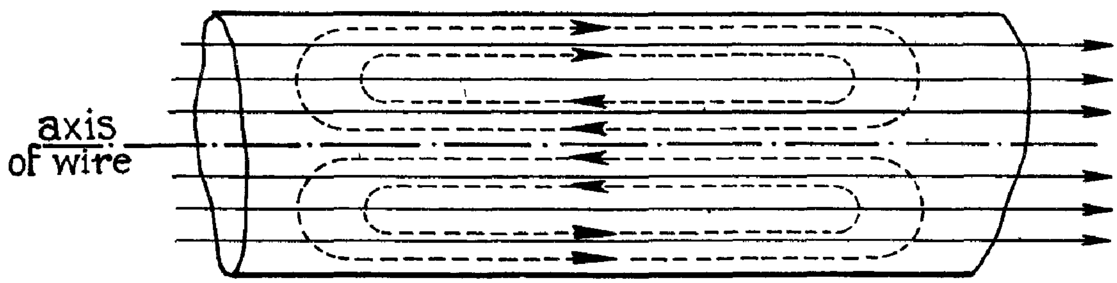


FIG. 13, CHAP. V.—Skin effect in isolated conductor.

how this increase occurs, imagine a conductor of D.C. resistance R ohms, of square cross-section and one inch side, to carry a current of I amperes. Divide the cross-section into four equal portions; each quarter of the conductor will then have a resistance of $4R$ ohms and with a uniform current distribution will carry $\frac{I}{4}$ amperes. The power dissipated in the conductor will therefore be $4 \times \left(\frac{I}{4}\right)^2 \times 4R = I^2 R$ watts. Now suppose that owing to some peculiar phenomenon, one of the quarters is prevented from carrying current, but that the total current remains as before, namely I amperes. Each remaining quarter must therefore carry $\frac{I}{3}$ amperes, and the power dissipated will be $3 \times \left(\frac{I}{3}\right)^2 \times 4R = \frac{4}{3} I^2 R$ watts.

By the definition given above the total effective resistance is the power dissipated divided by the square of the current, or $\frac{4}{3} I^2R \div I^2 = \frac{4}{3}R$, and the A.C. resistance of the conductor in this particular instance is $\frac{4}{3}$ times the D.C. resistance. If several conductors are situated in close proximity, as in an inductively wound coil, each turn will be the seat of eddy currents set up by the current in adjacent turns as well as by the current in the turn itself. The current is then constrained to flow in paths having the form shown by the shaded areas in fig. 14, and the resistance is still further increased. This is the proximity effect mentioned above.

The total copper loss in a conductor may be obtained by adding the losses due to skin and proximity effects to the ordinary D.C. loss. Considering only the resistance of wires of circular cross-section it can be shown that the increase of resistance is proportional to a factor $z = \pi d \sqrt{\frac{2f\mu}{\varrho}}$ where d is the diameter of the wire in centimetres, f the frequency, μ the permeability of the material and ϱ its specific resistance in E.M. units.

The A.C. resistance of a straight wire remote from all other conductors is

$$R_s = R_{dc}\,(1 + F)$$

where F is the skin effect factor. If z is less than 2, $F = \frac{z^4}{192}$, while if z is greater than 100, F approaches the value $\frac{\sqrt{2}\,z - 3}{4}$. Table VIII, Appendix A gives the value of F for the range $z = 0{\cdot}1$ to $z = 100$.

The energy loss due to proximity effect may also be calculated and added to the other components to give the total energy loss. Taking the simplest case of two parallel wires of

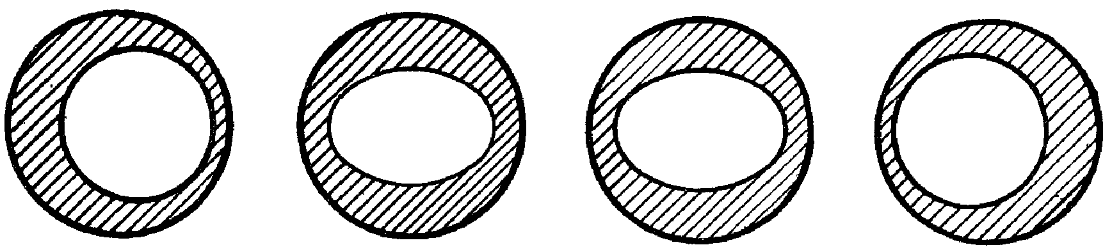


FIG. 14, CHAP. V.—Distribution of current in adjacent conductors (proximity effect).

diameter d centimetres the axes of which are separated by a distance of c centimetres, the additional resistance R_h due to proximity effect is

$$R_h = R_{dc}\,\frac{G\,d^2}{c^2}$$

where G, the proximity factor, also depends upon the value of z. If z is less than $0{\cdot}5$, $G = \frac{z^4}{64}$ approximately, while if z is greater than 100, $G = \frac{\sqrt{2}\,z - 1}{8}$. The total A.C. resistance then becomes $R_{ac} = R_{dc}\left(1 + F + G\frac{d^2}{c^2}\right)$ provided that the ratio d/c does not approach unity closely. If however the wires are very close together a further correction must be applied in the form of a factor J and

$$R_{ac} = R_{dc}\left(1 + F + \frac{G\frac{d^2}{c^2}}{1 - J\frac{d^2}{c^2}}\right)$$

This formula may also be extended to apply to any number of spaced parallel wires, becoming

$$R_{ac} = R_{dc}\left(1 + F + \frac{u\,G\frac{d^2}{c^2}}{1 - J\frac{d^2}{c^2}}\right)$$

where u depends upon the number of wires in parallel. Values of G, J and u are given in Tables IX and X, Appendix A.

13. The above expressions may be found to be of practical value in calculating the resistance of aerial wires and feeder lines, or of solenoids or flat spiral coils, provided they are wound in a single layer and that the coil radius is large compared with the winding length or depth. In other cases the calculation of the proximity effect becomes extremely complicated and only some practical conclusions can be given.

(i) Both the skin effect and proximity effect would be zero if equal current distribution could be achieved. An approximation to equal current distribution can be attained by using cable consisting of very fine strands of wire so plaited or twisted that every wire occupies successively a position near the centre and on the outside, each wire being thus of equal inductance and resistance. The strands may be made up in groups of three, similar to a rope, and it is found that there is an optimum value for the diameter of the constituent wires, depending upon the frequency. At extremely high frequencies it is impossible in practice to approach the extremely fine stranding demanded by theory, and a solid wire offers less resistance than a stranded one of incorrect design. To obtain any benefit from the use of " litz " cable, as this kind of wire is known, a very high standard of workmanship is desirable. A broken strand, or a single strand badly soldered to the terminal of the coil, may cause the latter to be less efficient than an ordinary coil of solid wire. In the repair of transmitting inductances and similar apparatus this should be constantly borne in mind.

(ii) If solid wire is used, e.g. for receiving inductances, there is an optimum gauge of wire for any given coil dimensions and frequency. As an example, consider the dimensions shown in fig. 15 which were chosen for a coil of 2,000 microhenries intended for reception of Droitwich

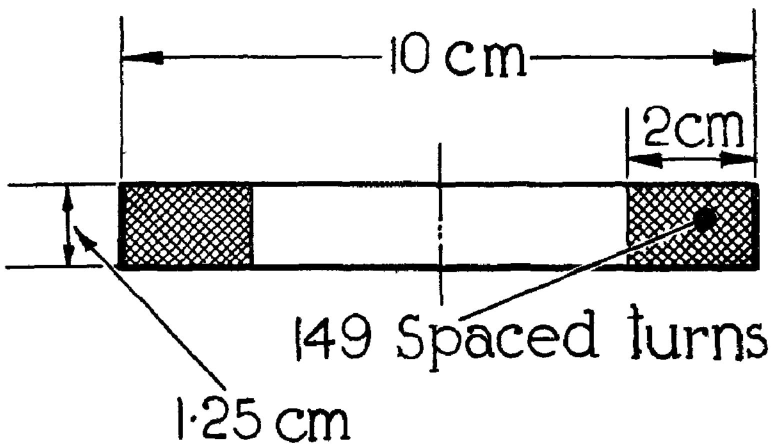


FIG. 15, CHAP. V.—Dimensions of multi-layer coil, 2,000 μ H.

(200 kc/s). Different sizes of wire were used for a number of windings, the winding space being completely filled on each occasion by spacing the turns as much as possible. It was found that if wound with the requisite number of turns of 36 s.w.g. its total resistance at the given frequency was 22·9 ohms. This was reduced to 15·6 ohms by rewinding with 32 s.w.g. but a further increase of wire diameter increased the resistance to 19·7 ohms. The greater portion of this increase is due to the proximity effect. An increase of wire gauge to 24 s.w.g. increased the resistance to 40 ohms.

(iii) Where heavy high-frequency currents are to be carried, copper tube is as efficient as solid copper, and if this is not available copper strip should be employed. Copper tube is the only conductor suitable for transmitting inductances for frequencies above 3,000 kc/s, and is preferably silver-plated, particularly if rubbing contacts are to be employed. The silver plating prevents oxidisation and great care should be taken not to destroy it by harsh cleaning processes.

Resistance in an A.C. circuit

14. It is often convenient to consider the case where any of the three properties, inductance, capacitance or resistance, is entirely absent from a given circuit, for in practice the effects of one

or more of these is often negligible. Suppose therefore that a certain circuit is entirely devoid of capacitive and inductive effects, and consider the application of an E.M.F. $e = \mathscr{E} \sin \omega t$ to a resistance of R ohms. As there is no other E.M.F. (such as a counter-E.M.F. of self-induction) and no tendency for electrons to accumulate in any part of the circuit, i.e. no capacitance, a current will commence to flow as soon as the E.M.F. is applied, and this current will be proportional to the E.M.F. Thus in a circuit of this nature Ohm's law is obeyed, and the instantaneous value of the current is $\frac{e}{R}$ or $\frac{\mathscr{E} \sin \omega t}{R}$. The peak value of the current is $\mathscr{I} = \frac{\mathscr{E}}{R}$ and its effective or R.M.S. value $I = \frac{E}{R}$. The current will be in phase with the applied E.M.F. as shown by curves and vectors in fig. 16.

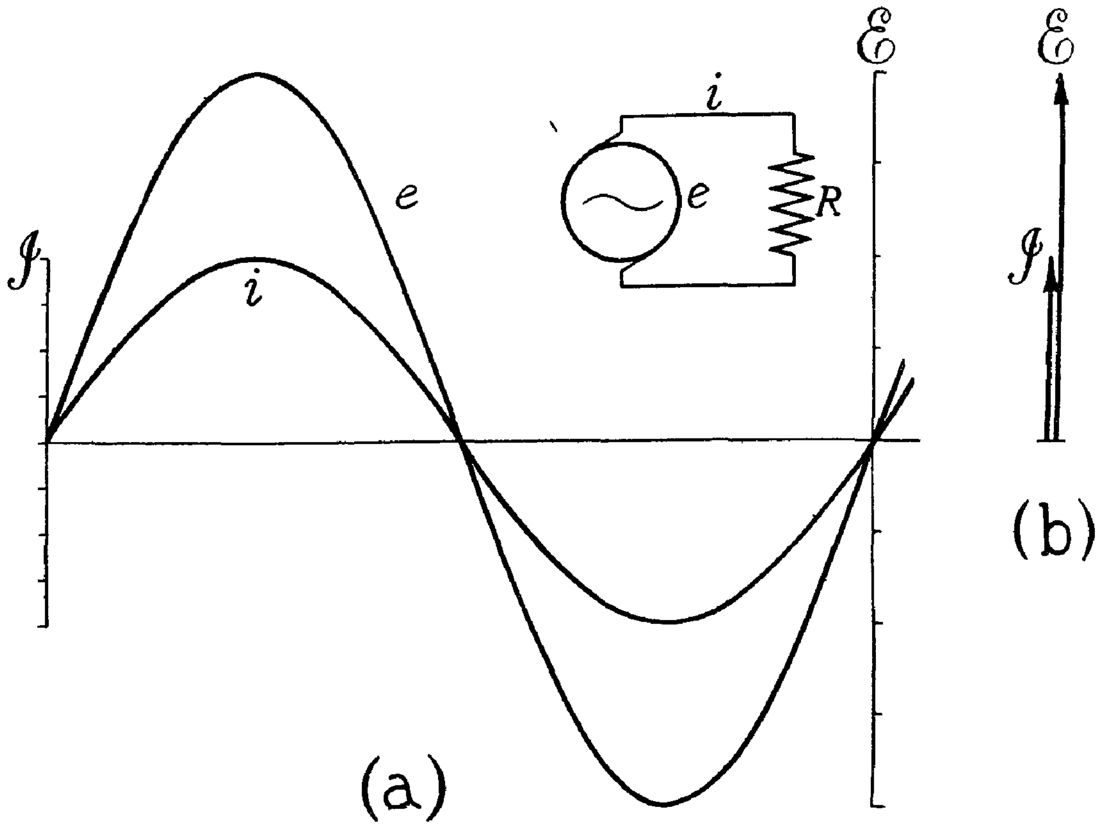


FIG. 16, CHAP. V.—Effect of resistance in A.C. circuit.

Inductance

15. The inductance of a given circuit is not truly a constant, but depends to some extent upon the frequency and intensity of the current flowing in it. This is particularly true if any ferro-magnetic substance is situated within the field of the coil. It has been shown that the value of the inductance is given by an equation having the form $L = \frac{KN^2}{S}$ where S is the reluctance of the path of the tubes of magnetic flux. When iron or any other ferro-magnetic material is present in the magnetic field, the reluctance varies with the factors above-mentioned, because both affect the effective permeability of the magnetic path. It is usual to distinguish between inductances designed for low frequencies, below about 10,000 cycles per second, and high frequencies which are those higher than 10,000 cycles per second. This dividing line between high and low frequencies cannot be drawn with precision and for some purposes it is desirable

to include frequencies up to about 20,000 cycles per second in the " low frequency " category. This division corresponds to the fact that vibrations of the air of from 16 up to about 20,000 cycles per second produce the sensation of sound, and it is usual to refer to this range of frequency as the audio-frequency band even if no question of sound production is involved. The frequencies above the audible limit are referred to in a corresponding manner as radio-frequencies even if no question of radiation arises.

Inductances intended for use in circuits carrying audio-frequency currents are invariably fitted with iron cores, which are laminated to reduce eddy current loss, while the iron used is of a kind which has low hysteresis loss. In the audio-frequency portions of radio apparatus, conditions frequently arise in which it is necessary that the winding should carry a current of complex wave-form, consisting of a steady component and several alternating components. The inductance of such a coil under these circumstances depends upon the magnitudes of both the A.C. and D.C. components. To illustrate this, consider the magnetisation or B/H curve, shown in fig. 17. Suppose that the magnetising force due to the D.C. is that corresponding to

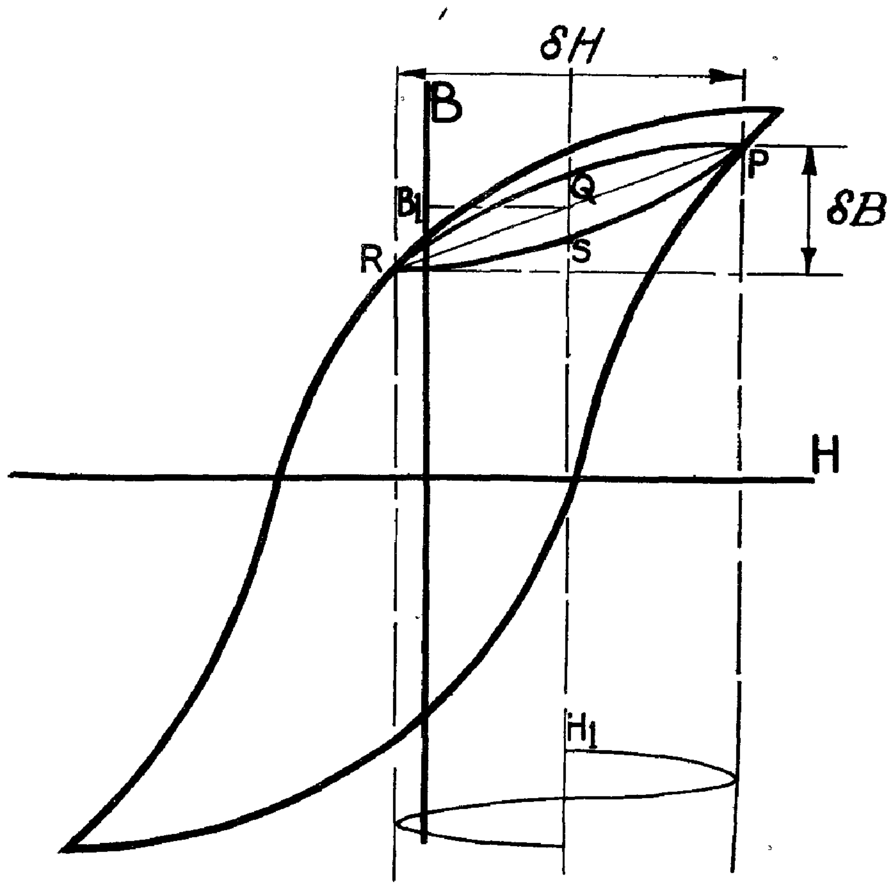


FIG. 17, CHAP. V.—Hysteresis loop showing definition of incremental permeability.

the ordinate H_1, then the average flux density in the core will be B_1. The magnetising force due to the A.C. will be superimposed upon this, and the iron will be carried through a cycle of magnetisation, as shown by the small hysteresis loop SPQR. The effective permeability with respect to the alternating current is then not the ratio B_1/H_1 as for direct current, but the average slope of the small loop, which may be represented by $\frac{\delta B}{\delta H}$, and this is always smaller than B_1/H_1, hence the inductance of the coil is reduced by the presence of the direct current. The larger the steady magnetising force the less will be the value of $\frac{\delta B}{\delta H}$. The expression $\frac{\delta B}{\delta H}$ is called the incremental permeability to distinguish it from the ordinary permeability B/H. The introduction of an air gap into the iron core results in an increase in reluctance of the magnetic path, but also an increase in the incremental permeability. The net effect may be an increase in the effective inductance for alternating current while the inductance will also tend to remain constant, hence the cores of inductances for use under conditions in which the conductor carries both D.C. and A.C. almost invariably contain a small air gap.

16. Except in special circumstances, iron cores are not used in inductances designed for radio-frequency circuits for two reasons. First, the eddy current loss can only be kept within reasonable limits by sub-division of the core material (i.e. lamination) to a degree which is often impracticable. Second, it is desirable that the value of the inductance shall be independent of both the frequency and magnitude of the current. Radio-frequency circuits therefore usually employ air-core inductances, typical specimens of which have been briefly described in Chapter II. The capacitance which exists between all adjacent conductors must also be taken into account, for it is obvious that the presence of any metal inside the coil will increase this capacitance, whereas is it usually desirable to maintain it at the lowest possible value.

In spite of these difficulties, however, the cores of certain inductances in radio-frequency circuits are of iron or nickel-iron alloy in the form of a very fine powder which is amalgamated with a phenolic material as a binder. These coils are largely used in aircraft receivers, and receive further mention in a later chapter.

Inductance in an A.C. circuit

17. The effect of inductance in any circuit is to oppose any change in the value of the current, owing to the counter-E.M.F. set up by the changing flux linkage. If the inductance of the circuit is L henries, Faraday's and Lenz's laws tell us that the counter-E.M.F. is $l_b = -L\frac{di}{dt}$ volts. Now let us assume that an alternating current $i = \mathcal{I} \sin \omega t$ flows in a circuit having an inductance of L henries, but of negligible resistance and capacitance. Earlier in this chapter it was shown that if the current is of the assumed form, the rate of change of current is $\frac{di}{dt} = \omega \mathcal{I} \cos \omega t$. The counter-E.M.F. of self induction is therefore $-\omega L \mathcal{I} \cos \omega t$.

The applied E.M.F. must be equal and opposite to this counter-E.M.F. and no more, since the only work which the E.M.F. has to perform is to maintain a sinusoidal flux in the inductance. The applied E.M.F. is therefore

$$e = \omega L \mathcal{I} \cos \omega t$$

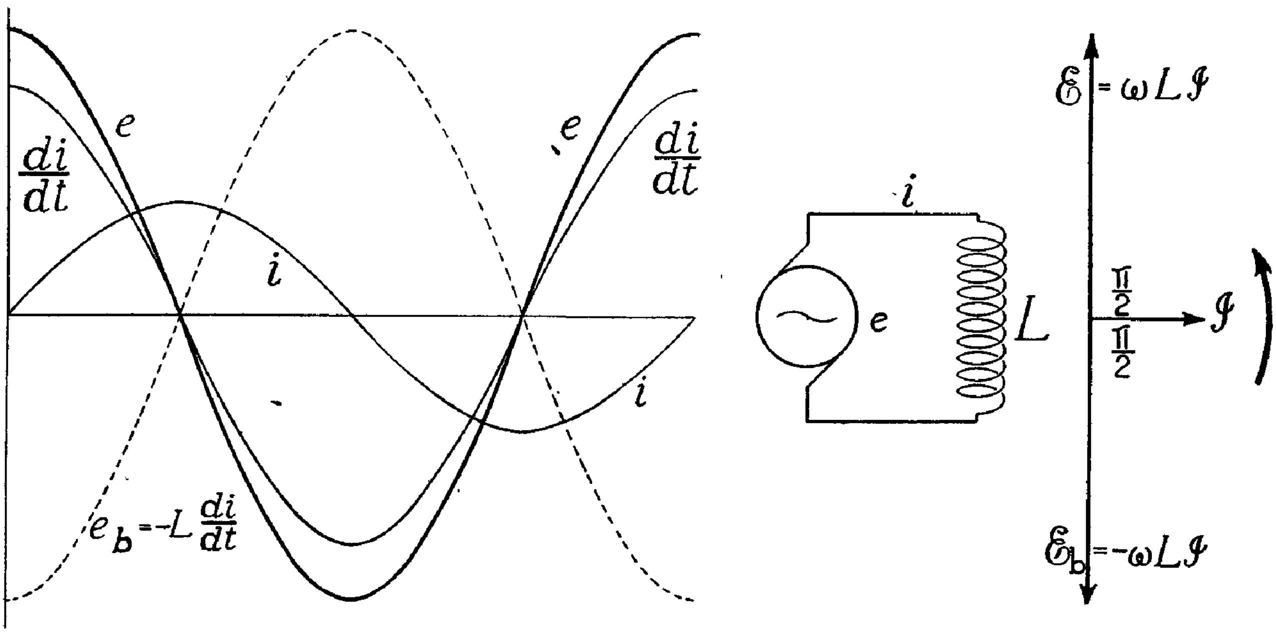
$$= \omega L \mathcal{I} \sin\left(\omega t + \frac{\pi}{2}\right)$$

Hence the applied E.M.F. is also sinusoidal in form, but leads on the current by $\frac{\pi}{2}$ radians or 90 degrees. The frequency of the applied E.M.F. is the same as that of the current, which is of course to be expected. The peak value $\mathcal{E}$ of the E.M.F. is $\omega L \mathcal{I}$ volts,and the phase relationship between $\mathcal{E}$ and $\mathcal{I}$ is shown by curves and vectors in fig. 18. It will be observed that this relation may be expressed either as "$\mathcal{E}$ leading on $\mathcal{I}$" or "$\mathcal{I}$ lagging behind $\mathcal{E}$". The latter is more usual, and the relation between the current and voltage may be written

$$i = \frac{\mathcal{E}}{\omega L} \sin\left(\omega t - \frac{\pi}{2}\right)$$

It is often convenient to consider R.M.S. rather than instantaneous values. Since $I = \dfrac{\mathscr{I}}{\sqrt{2}}$ and $E = \dfrac{\mathscr{E}}{\sqrt{2}}$, the relation between I and E is given by the equation $I = \dfrac{E}{\omega L}$. Comparing this with Ohm's law for direct current $I = \dfrac{E}{R}$, it will be seen that ωL is analogous with R in deciding the magnitude of the current for a given applied voltage.

Although L itself is in henries, the expression ωL is in ohms, because the dimensions or physical attributes of the ratio of voltage to current must always be the same. The factor ωL is termed inductive reactance of the circuit, and may be thought of as the opposition offered by an inductance to the flow of an alternating current, but it must be borne in mind that in addition to limiting the magnitude of the current, it causes the current to lag behind the E.M.F. which produces it. The symbol for reactance is X, and inductive reactance is denoted by X_L thus, $X_L = \omega L$.



(a) (b)

FIG. 18, CHAP. V.—Effect of inductance in A.C. circuit.

Inductances in series and parallel

18. If the alternating current $i = \mathscr{I} \sin \omega t$ flows through several inductances L_1, L_2, L_3, etc., in succession, a counter-E.M.F. is set up in each inductance, equal to $-\omega L_1 i$, $-\omega L_2 i$, $-\omega L_3 i$, etc. The applied voltage must be sufficient to overcome the sum of all these counter-E.M.F's. and therefore, considering R.M.S. values only,

$$\begin{aligned} E &= \omega L_1 I + \omega L_2 I + \omega L_3 I, \text{ etc.} \\ &= \omega I\ (L_1 + L_2 + L_3 \ldots\ldots\ldots) \\ &= \omega L I \end{aligned}$$

where $\quad L = L_1 + L_2 + L_3.$

Hence the total inductance L is the sum of the individual inductances which are in series in the circuit.

If the inductances L_1, L_2, L_3, etc., are placed in parallel, having a common sinusoidal P.D. of V volts between their ends, the current in each inductance, taking R.M.S. values as before, will be

$$I_1 = \frac{V}{\omega L_1}$$

$$I_2 = \frac{V}{\omega L_2}$$

$$I_3 = \frac{V}{\omega L_3}$$

$$I_1 + I_2 + I_3 = \frac{V}{\omega}\left(\frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3}\right)$$

$I_1 + I_2 + I_3$ is the total current flowing in the main or unbranched portion of the circuit and may be denoted by I. I and V are connected by the relation $I = \frac{V}{\omega L}$, where L is the joint inductance of L_1, L_2, L_3 in parallel.

Collecting these equations we see that

$$I = \frac{V}{\omega L}$$

$$I_1 + I_2 + I_3 = \frac{V}{\omega}\left(\frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3}\right)$$

since $\quad I = I_1 + I_2 + I_3$

$$\frac{1}{L} = \frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3}$$

The reciprocal of the joint inductance is equal to the sum of the reciprocals of the individual inductances. It will be observed that the rules for inductances in series and in parallel are the same as for resistances.

A particular instance which often arises in practice is the calculation of the joint inductance of two inductances L_1 and L_2 in parallel. Since

$$\frac{1}{L} = \frac{1}{L_1} + \frac{1}{L_2}$$

giving the right-hand member of the equation a common denominator

$$\frac{1}{L} = \frac{L_1 + L_2}{L_1 L_2}$$

whence $\quad L = \dfrac{L_1 L_2}{L_1 + L_2}$

and it is seen that the joint inductance of two inductances in parallel is given by the product of the two divided by their sum.

Effect of mutual inductance between coils in parallel

19. Consider the circuit given in fig. 19 which may represent the two coils of a variometer inductance. If an R.M.S. voltage E of frequency $\frac{\omega}{2\pi}$ is applied, a current of I_1 amperes will flow in the coil L_1 and a current of I_2 amperes in the coil L_2. If the mutual inductance between the coils is zero the counter-E.M.F. induced in the two windings will be $-\ \omega L_1 I_1$ and $-\ \omega L_2 I_2$ volts respectively. If the mutual inductance has the finite value M henries, however, the

counter-E.M.F. induced in the coil L_1 will be $-(\omega L_1 I_1 + \omega M I_2)$ volts and in the coil L_2, $-(\omega L_2 I_2 + \omega M I_1)$ volts. As the resistance is assumed to be negligible, the counter-E.M.F. is equal and opposite to the applied E.M.F. and

$$\omega L_1 I_1 + \omega M I_2 = E \quad \ldots \quad (a)$$

$$\omega M I_1 + \omega L_2 I_2 = E \quad \ldots \quad (b)$$

$$\text{or } \omega L_1 L_2 I_1 + \omega M L_2 I_2 = L_2 E \quad \ldots \quad (c)$$

$$\omega M^2 I_1 + \omega M L_2 I_2 = M E \quad \ldots \quad (d)$$

Subtracting (a) from (c)

$$\omega(L_1 L_2 - M^2) I_1 = (L_2 - M) E \quad \ldots \quad (e)$$

$$\text{also } \omega L_1 M I_1 + \omega M^2 I_2 = M E \quad \ldots \quad (f)$$

$$\omega L_1 M I_1 + \omega L_1 L_2 I_2 = L_1 E \quad \ldots \quad (g)$$

Subtracting (f) from (g)

$$\omega(L_1 L_2 - M^2) I_2 = (L_1 - M) E \quad \ldots \quad (h)$$

From (e) and (h)

$$I_1 = \frac{(L_2 - M) E}{\omega(L_1 L_2 - M^2)}; \quad I_2 = \frac{(L_1 - M) E}{\omega(L_1 L_2 - M^2)}.$$

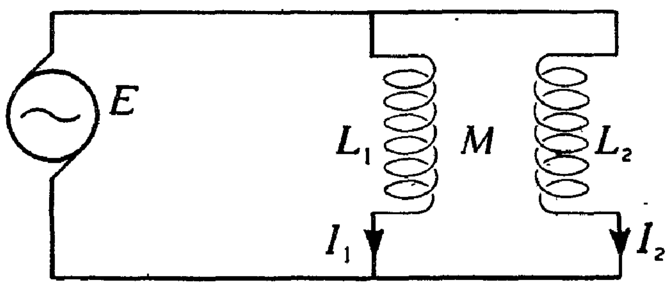


FIG. 19, CHAP. V.—Inductances in parallel, possessing mutual inductance.

The total current I is $I_1 + I_2$

$$I = \frac{L_1 + L_2 - 2M}{\omega(L_1 L_2 - M^2)} E$$

The effective reactance of the two coils in parallel is $\frac{E}{I} = \omega L$, and

$$\omega L = \frac{\omega(L_1 L_2 - M^2)}{L_1 + L_2 - 2M}$$

Hence the effective inductance of the parallel combination, including the mutual inductance, is

$$L = \frac{L_1 L_2 - M^2}{L_1 + L_2 - 2M}$$

When the coils are perpendicular to each other the mutual inductance is zero and the effective inductance is $\frac{L_1 L_2}{L_1 + L_2}$ as already shown. With any other relative disposition the mutual inductance has a finite value which may be either positive or negative. These signs are purely conventional, and it is convenient to regard the mutual inductance as positive if its effect is to increase the total inductance; this was adopted in writing the equations (a) and (b). The opposite signs are sometimes adopted in certain theoretical work. The effect of the mutual upon the total effective inductance is seen in the following example.

Example 3.—(i) The two coils of a variometer, each having an inductance of 100 microhenries, are connected in parallel. When the coils are co-axial, the mutual inductance is 90 microhenries. Find the maximum and minimum inductance.

When M is positive, $L = \dfrac{L_1 L_2 - M^2}{L_1 + L_2 - 2M}$

$$= \frac{L_1^2 - M^2}{2(L_1 - M)}$$

$$= \frac{(L_1 + M)(L_1 - M)}{2(L_1 - M)}$$

$$= \frac{L_1 + M}{2} = \frac{190}{2} = 95\mu H$$

When M is negative, $L = \dfrac{(L_1 + M)(L_2 - M)}{2(L_1 + M)}$

$$= \frac{L_1 - M}{2} = 5\ \mu H$$

(ii) If the mutual inductance were 98 microhenries, what would be the total range of inductance ?

From the above, with positive M, $L = \dfrac{100 + 98}{2} = 99\ \mu H$

and with negative M. $L = \dfrac{100 - 98}{-2} = 1\ \mu H$

Hence the total range is from 99 to 1 microhenry. If every tube of magnetic flux linked with every turn of both coils, the inductance range would be from 100 to 0 microhenries.

Resistance and inductance in series

20. In the circuit diagram of fig. 20 a source of alternating E.M.F. supplies current to a circuit consisting of an inductance of L henries and a resistance of R ohms, connected in series. It is required to find the current which will flow, and the relative phases of current and voltage. The opposition offered by the circuit is now of two kinds (i) the resistance, which limits the value of the current, but will cause no phase difference, and (ii) the inductive reactance, which also limits the value of the current, and tends to cause it to lag behind the applied E.M.F. by $\frac{\pi}{2}$ radians. The applied E.M.F. can therefore be divided into two components, one of which may be considered to overcome the resistance, or to supply the energy converted into heat; and the other to overcome the inductive reactance or to supply energy which is stored in the form of a magnetic field when the current is increasing in value, and returned to the circuit when the current is decreasing.

The instantaneous values of these components may be denoted by v_R and v_L. Assuming the current to be sinusoidal, the curves in fig. 20 show the nature of their variation. The component v_R has the instantaneous value iR, and is in phase with the current, while the component v_L has the instantaneous value ωLi, and leads on the current by $\frac{\pi}{2}$ radians or 90°. The total voltage supplied by the alternator at any instant is found by adding the ordinates of the two curves, giving the resultant curve e, which represents the applied voltage. It is seen that the latter reaches its maximum value before the instant of maximum current, but that the angle of phase difference is less than 90°.

The relations between the maximum values $\mathcal{V}_R$, $\mathcal{V}_L$ and $\mathcal{E}$ are more rapidly obtained by a vector diagram (*see* fig. 20). The component $\mathcal{V}_R$ is in phase with $\mathcal{I}$ and the component $\mathcal{V}_L$ leads on $\mathcal{I}$ by 90°. The peak value of the applied E.M.F. is the vector sum of these and is obtained by the method previously described. Their sum is however easily obtained by the well-known "theorem of Pythagoras", i.e. if a and b are the two shorter sides of a right-angled triangle the length of the third side is $\sqrt{a^2 + b^2}$. Applying this theorem

$$\mathcal{E} = \sqrt{\mathcal{V}_R^{\,2} + \mathcal{V}_L^{\,2}}$$

Now $$\mathcal{V}_R = R\,\mathcal{I},\ \mathcal{V}_L = \omega L\,\mathcal{I}$$

and $$\mathcal{E} = \sqrt{(R\,\mathcal{I})^2 + (\omega L\,\mathcal{I})^2}$$

$$= \mathcal{I}\sqrt{R^2 + (\omega L)^2}$$

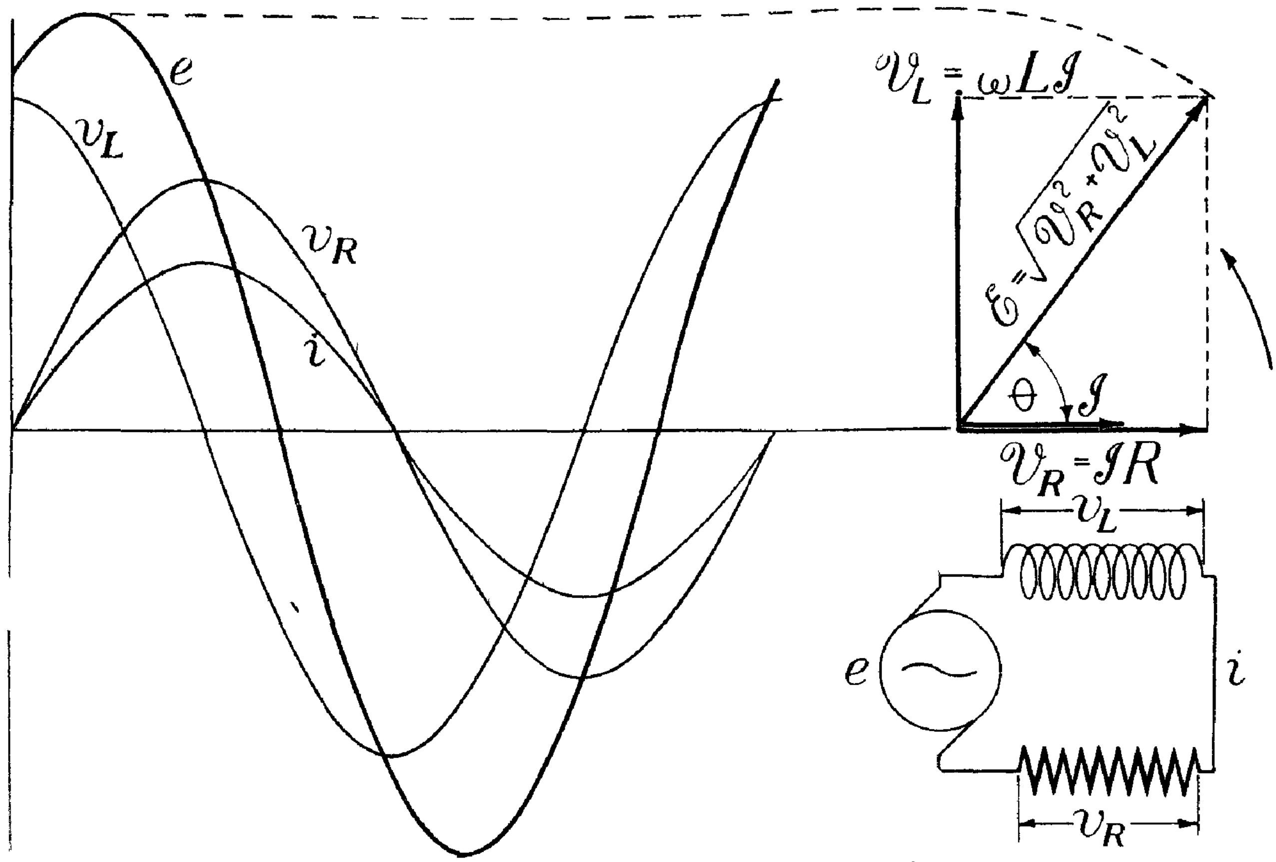


FIG. 20, CHAP. V.—Effect of inductance and resistance in series.

From the vector diagram it is obvious that the current lags upon the applied voltage by an angle θ. The magnitude of this angle is found from either of the following formulae :—

$$\tan\theta = \frac{\omega L\,\mathcal{I}}{R\,\mathcal{I}} = \frac{\omega L}{R},$$

$$\sin\theta = \frac{\omega L}{\sqrt{R^2 + (\omega L)^2}} = \frac{\omega L}{Z}$$

$$\cos\theta = \frac{R}{\sqrt{R^2 + (\omega L)^2}} = \frac{R}{Z}$$

The instantaneous value of the current is

$$i = \frac{\mathcal{E}}{\sqrt{R^2 + (\omega L)^2}} \sin(\omega t - \theta)$$

where θ is the angle whose tangent is $\frac{\omega L}{R}$ (*see* above). The notation usually used is

$$\theta = \tan^{-1} \frac{\omega L}{R}$$

The R.M.S. value of the current is

$$I = \frac{E}{\sqrt{R^2 + (\omega L)^2}}$$

which may again, be compared with Ohm's law. In this case $\frac{E}{I} = \sqrt{R^2 + (\omega L)^2}$, and $\sqrt{R^2 + (\omega L)^2}$ is in ohms. It may be considered to represent the total opposition of the current to the flow of current and is called the impedance of the circuit. The symbol Z is used for impedance when it is not necessary or possible to express it in a more detailed form.

Example 4.—An alternating E.M.F. of 220 volts peak value, having a frequency of 100 cycles per second, is applied to an inductance of 1·5 henries and a resistance of 600 ohms in series. Find (i) the R.M.S. value of the current; (ii) the peak P.D. at the terminals of the inductance; (iii) the angle of phase difference.

$$\omega = 2\pi f = 628$$

$$\omega L = 628 \times 1{\cdot}5 = 942 \text{ ohms}$$

$$Z = \sqrt{R^2 + (\omega L)^2} = \sqrt{600^2 + 942^2}$$

$$= \sqrt{36 \times 10^4 + 88{\cdot}75 \times 10^4}$$

$$= 100\sqrt{124{\cdot}75}$$

$$= 1120 \text{ ohms approx.}$$

$$E = \frac{\mathcal{E}}{\sqrt{2}} = {\cdot}707 \times 220 = 155{\cdot}5 \text{ volts.}$$

(i) $$I = \frac{E}{Z} = \frac{155{\cdot}5}{1120} = {\cdot}139 \text{ amperes (nearly).}$$

(ii) Peak P.D. at inductance terminals $= \omega L \mathcal{I}$ volts

$$\mathcal{I} = \sqrt{2}\, I = 1{\cdot}414 \times {\cdot}139 = 1{\cdot}965 \text{ amperes}$$

or $$\mathcal{I} = \frac{\mathcal{E}}{Z} = \frac{220}{1120} \text{ amperes} = 1{\cdot}965 \text{ amperes}$$

$$\omega L \mathcal{I} = 942 \times 1{\cdot}965 = 185 \text{ volts.}$$

(iii) $$\theta = \tan^{-1} \frac{\omega L}{R}$$

$$= \tan^{-1} \frac{942}{600}$$

$$= \tan^{-1} 1{\cdot}57$$

Reference to a table of tangents gives $\theta = 58°$, to the nearest degree, which is of sufficient accuracy for all practical work. Since the reactance is inductive, the current will lag on the applied E.M.F. by 58°.

Impedances in series

21. If a number of pieces of apparatus having both resistance and inductance are connected in series and an alternating E.M.F. is applied, the resulting current, $\mathcal{I}$ *sin* ωt, will set up a P.D. between the terminals of each instrument. If the resistance of each is denoted by R_1, R_2, R_3, etc., and the corresponding inductance by L_1, L_2, L_3, etc., the peak P.D.'s will be $\mathcal{V}_1 = \mathcal{I}\sqrt{R_1^2 + (\omega L_1)^2}$ $\mathcal{V}_2 = \mathcal{I}\sqrt{R_2^2 + (\omega L_2)^2}$, etc.

$\mathcal{I}$ will lag on $\mathcal{V}_1$, by an angle $\tan^{-1}\frac{\omega L_1}{R_1}$, on $\mathcal{V}_2$ by an angle $\tan^{-1}\frac{\omega L_2}{R_2}$, etc. The peak value, $\mathcal{E}$, of the applied E.M.F. will be equal to the vector sum of $\mathcal{V}_1$, $\mathcal{V}_2$, $\mathcal{V}_3$, and the resulting vector diagram is shown in fig. 21. It will be seen that

$$\mathcal{E} = \sqrt{\mathcal{I}^2 (R_1 + R_2 + R_3)^2 + \mathcal{I}^2 (\omega L_1 + \omega L_2 + \omega L_3)^2}$$

$$\text{or } \mathcal{E} = \mathcal{I}\sqrt{(R_1 + R_2 + R_3)^2 + \omega^2 (L_1 + L_2 + L_3)^2}$$

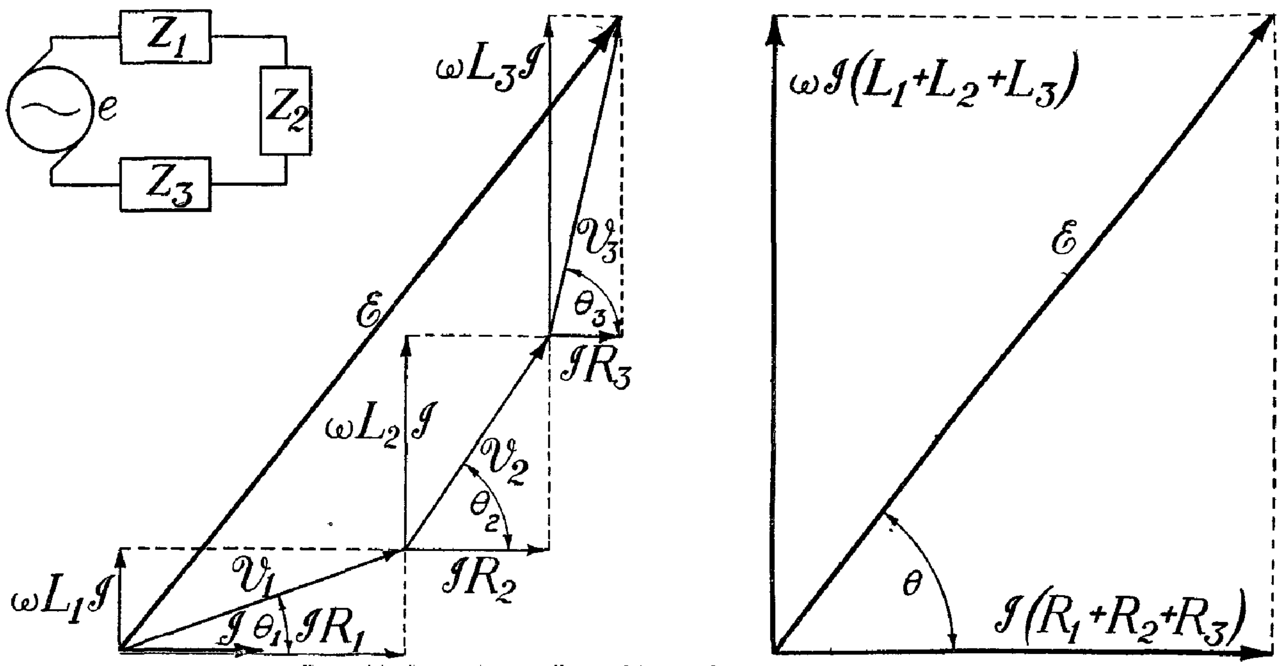


FIG. 21, CHAP. V.—Effect of inductive impedances in series.

The current $\mathcal{I}$ will lag on the applied E.M.F. $\mathcal{E}$ by an angle $\tan^{-1}\frac{\omega(L_1 + L_2 + L_3)}{R_1 + R_2 + R_3}$, and this is true no matter how many separate impedances of this kind are connected in series.

In any of the foregoing vector diagrams, the three vectors form the sides of a right-angled triangle having sides $\mathcal{I}Z$, $\mathcal{I}X$, $\mathcal{I}R$. The relative lengths of these sides, that is the shape of the triangle, will not be altered if each side is divided by $\mathcal{I}$, giving a triangle of sides Z, X, R. This is called the impedance triangle of the circuit and may be used instead of the true vector diagram, in order to obtain the angle of phase difference θ, by means of the equations $\tan\theta = \frac{X}{R}$, $\sin\theta = \frac{X}{Z}$, $\cos\theta = \frac{R}{Z}$

Capacitance

22. In Chapter I the flow of electric current in a circuit was compared with the flow of water in a pipe line. Let us consider this analogy further. In fig. 22 the pipe line is supplied

from a pump which moves the water to and fro round the circuit instead of continuously in the same direction. The friction of the water against the sides of the pipe may be considered to represent the resistance of the electrical circuit, and the inertia of the water, that is its opposition to a change of motion, to represent its inductance. Suppose that the flow of water in the pipe line is restricted at one point by a flexible diaphragm, then provided that the pressure is insufficient

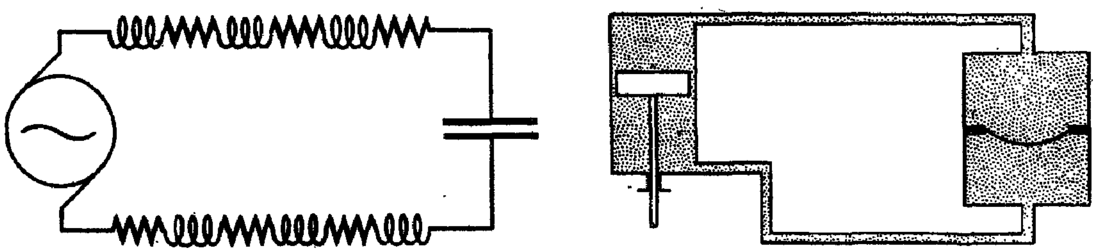


FIG. 22, CHAP. V.—Hydraulic analogue of A.C. circuit containing a condenser.

to burst this diaphragm, the pump can still move the water to and fro in the line, the diaphragm being stretched in one direction and the other alternately. Some work will be done by the pump in stretching the diaphragm, but this energy is stored in the diaphragm, and is expended in moving the water when the pressure of the pump is relaxed. To this extent therefore the diaphragm relieves the pump of an amount of work exactly equal to that which was expended in stretching it, and the action of the diaphragm is analogous to the presence of a condenser in an alternating current circuit.

Capacitance in an A.C. circuit

23. Let us now suppose that a condenser of capacitance C farads is connected directly to a source of alternating E.M.F. of peak value $\mathscr{E}$. This condenser will be presumed to have no energy losses and therefore no resistance, using the latter term in the extended sense applicable to A.C. theory. We have seen that the law connecting the capacitance C, the P.D., v, between its plates, and the charge q, is $q = Cv$, and this is applicable at all times, a change of P.D. being accompanied by a change of charge. If the condenser P.D. is caused by an applied voltage, the latter will always be equal to the P.D. but acting in the opposite direction, and as a result of this the condenser voltage is often referred to as the counter-E.M.F. of the condenser. The condenser voltage is not strictly an E.M.F. for by definition an E.M.F. only exists when energy of some other form undergoes conversion into electrical energy, and in the condenser the energy due to its charge is stored in electrical form.

When an alternating E.M.F. is applied to the condenser, the charge introduced into the latter will be directly proportional to the E.M.F. Hence if the applied E.M.F. is $e = \mathscr{E} \sin \omega t$ the instantaneous charge will be $q = C\mathscr{E} \sin \omega t$. The current flowing into, or out of the condenser is the rate at which electricity enters or leaves, and is measured in coulombs per second or amperes. This may be concisely expressed by saying that the current is the rate at which the charge is changing, and using the notation hitherto adopted for quantities which vary with time

$$i = \frac{dq}{dt}$$

Now $q = C\mathscr{E} \sin \omega t$ and from previous discussion it follows that the rate of change of the latter will follow a cosine law, hence

$$i = \omega C \mathscr{E} \cos \omega t$$

$$= \omega C \mathscr{E} \sin\left(\omega t + \frac{\pi}{2}\right)$$

and it is apparent that the charging current varies in magnitude in a similar manner to the applied E.M.F. but leads on the latter by $\frac{\pi}{2}$ radians or 90° (fig. 23). If it is desired to express the relationship between the E.M.F. and current in R.M.S. values, we may write $I = \omega CE$, which may again be compared with Ohm's law by rearranging in the form $I = E\,\frac{1}{\omega C}$. The denominator $\frac{1}{\omega C}$ which is analogous to the resistance in the true Ohm's law, is expressed in ohms, and is called the capacitive reactance of the condenser, the symbol X_c being sometimes used when it is unnecessary to introduce any reference to the frequency.

Condensers in series

24. Suppose that in a circuit to which is applied an E.M.F. $e = \mathscr{E} \sin \omega t$, we have a number of condensers C_1, C_2, C_3, etc., arranged in series. Then the capacitive reactance or opposition of each will be $\frac{1}{\omega C_1}, \frac{1}{\omega C_2}, \frac{1}{\omega C_3}$ respectively, and the peak value of P.D. across each will consequently

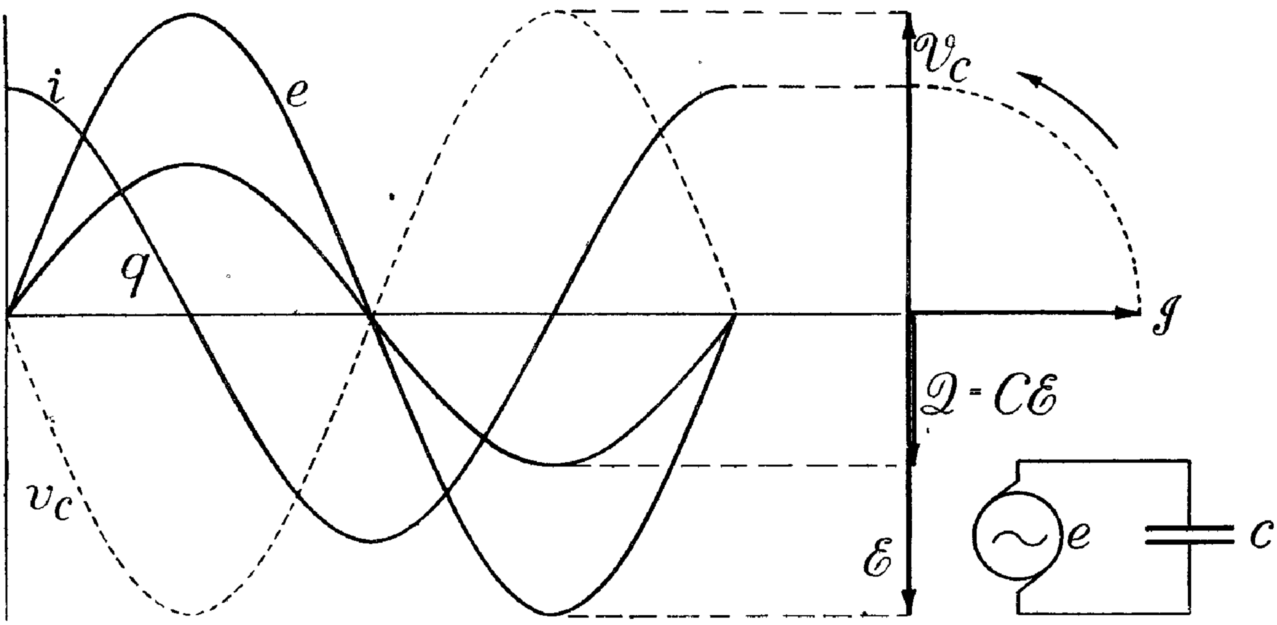


FIG. 23, CHAP. V.—Effect of capacitance in A.C. circuit.

be $\frac{\mathscr{I}}{\omega C_1}, \frac{\mathscr{I}}{\omega C_2}, \frac{\mathscr{I}}{\omega C_3}$. The total applied E.M.F. will be equal to the sum of the P.D's. Hence $\mathscr{E} = \mathscr{I}\left(\frac{1}{\omega C_1} + \frac{1}{\omega C_2} + \frac{1}{\omega C_3}\right)$. But if C is the total effective capacitance of the circuit

$$\mathscr{E} = \frac{\mathscr{I}}{\omega C}$$

and therefore $$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}$$

The effective capacitance of a number of condensers in series is therefore given by the reciprocal rule, just as for inductances or resistances in parallel.

When only two condensers are placed in series, a simpler formula is usually used.

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}$$

$$\frac{1}{C} = \frac{C_2 + C_1}{C_1\,C_2}$$

$$\therefore C = \frac{C_1\,C_2}{C_1 + C_2}$$

Capacitance and resistance in series

25. Just as inductance and resistance may be found in series in an A.C. circuit, so a circuit may contain capacitance and resistance in series. The opposition offered by the circuit will now be of two kinds, (i) the resistance, which will limit the current without causing any phase difference between $\mathscr{E}$ and $\mathscr{I}$, and (ii) the opposition caused by the counter-E.M.F. due to the charge in the condenser, which limits the current and also tends to cause the current to lead on the E.M.F. by 90°. If a sinusoidal current is caused to flow in such a circuit the peak value of the applied E.M.F. must be equal to the vector sum of the two P.D's, (i) between the ends of the resistance R, $\mathscr{V}_R$, and (ii) between the terminals of the condenser C, $\mathscr{V}_C$, respectively. If the peak value of the current is $\mathscr{I}$, $\mathscr{V}_R = \mathscr{I}R$ and $\mathscr{V}_C = \dfrac{\mathscr{I}}{\omega C}$, while $\mathscr{E} = \sqrt{\mathscr{V}_R^2 + \mathscr{V}_C^2}$.

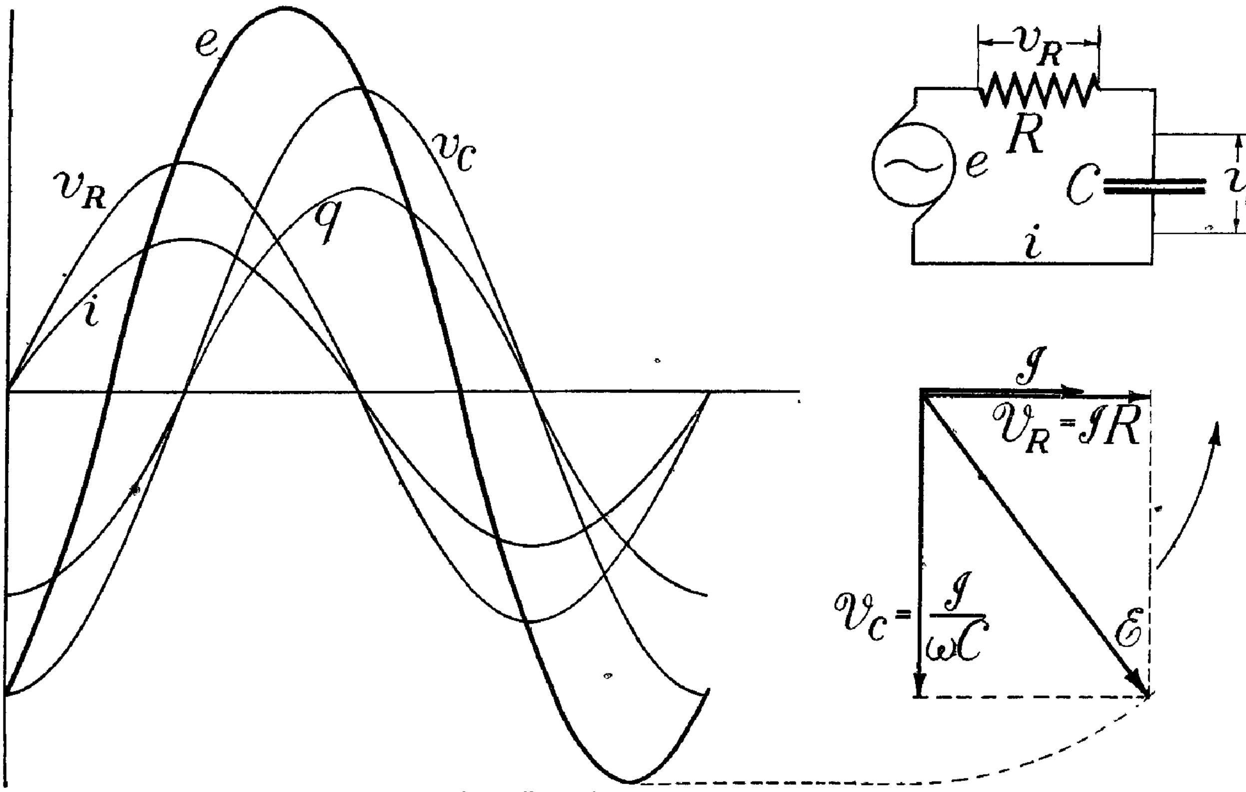


FIG. 24, CHAP. V.—Effect of capacitance and resistance in series.

$\mathscr{V}_R$ is in phase with $\mathscr{I}$, and $\mathscr{V}_C$ leads on $\mathscr{I}$ by 90°. The resulting phase relation between $\mathscr{E}$ and $\mathscr{I}$ is shown by the curves and vectors of fig. 24. With the aid of the vector diagram we deduce that

$$\mathscr{E} = \mathscr{I}\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}$$

and that $\mathscr{I}$ leads on $\mathscr{E}$ by an angle θ,

where $$\theta = tan^{-1}\frac{\frac{1}{\omega C}}{R} = \frac{1}{\omega CR}$$

The instantaneous value of the current is therefore

$$i = \frac{\mathscr{E}}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}} sin\ (\omega t + \theta)$$

In R.M.S. values, again

$$I = \frac{E}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}},$$

and the expression $\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2} = \sqrt{R^2 + X_c{}^2} = Z$

is the impedance of the circuit, in ohms.

Example 5.—A condenser of ·0015 microfarads and a resistance of 20 ohms are connected in series and an alternating E.M.F. of 1·5 millivolts R.M.S. at a frequency of 10^6 cycles per second is applied to the circuit. Find (i) the R.M.S. value of the resulting current (ii) the peak P.D. at the condenser terminals and (iii) the angle of phase difference.

$$\omega = 2\pi f = 2\pi \times 10^6$$

$$X_c = \frac{1}{\omega C} = \frac{1}{2\pi \times 10^6 \times \cdot 0015 \times 10^{-6}}$$

$$= \frac{10^3}{2 \times 1 \cdot 5}$$

$$= \frac{10^3}{3} = 106 \text{ ohms}$$

$$Z = \sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}$$

$$= \sqrt{20^2 + 106^2}$$

$$\doteqdot 108 \text{ ohms.}$$

(i) $$I = \frac{E}{Z} = \frac{1 \cdot 5}{108} \text{ milliamperes}$$

$$= \cdot 0139 \text{ milliamperes.}$$

(ii) $$\mathscr{V}_c = \frac{\mathscr{I}}{\omega C},\ \mathscr{I} = \sqrt{2}\, I = 1 \cdot 414 \times \cdot 0139 = \cdot 01965 \text{ milliamperes}$$

$$\therefore \mathscr{V}_c = 106 \times \cdot 01965 = 2 \cdot 08 \text{ millivolts.}$$

(iii) $$\theta = tan^{-1} \frac{X_c}{R}$$

$$= tan^{-1} \frac{106}{20}$$

$$= tan^{-1}\, 5 \cdot 3$$

$$\therefore \theta \doteqdot 80^\circ$$

i.e. $\mathscr{I}$ leads on $\mathscr{E}$ by 80°.

Inductance, capacitance and resistance in series

26. In fig. 25 is shown an alternating E.M.F. of peak value $\mathscr{E}$, applied to a circuit containing a resistance R, an inductance L, and a condenser C, in series. The alternator in this instance has to perform three duties ; it must supply (i) a voltage equal to the D.P. across the resistance, $\mathscr{V}_R = \mathscr{I}R$ (ii) a voltage equal to the counter-E.M.F. of the inductance, $\mathscr{V}_L = \omega L \mathscr{I}$ and (iii) a voltage equal to the counter-E.M.F. of the condenser, $\mathscr{V}_c = \frac{\mathscr{I}}{\omega C}$. The three components of the

total E.M.F. $\mathcal{E}$, are shown in the vector diagram, from which it will be seen that the vector $\mathcal{V}_R$ is in phase with the current, the vector $\mathcal{V}_L$ is 90° leading, and the vector $\mathcal{V}_C$ 90° lagging on the vector $\mathcal{I}$. This must be interpreted as signifying that the vectors $\mathcal{V}_C$ and $\mathcal{V}_L$ partly cancel each other, the P.D. of the condenser assisting, during certain portions of each cycle, to create or destroy the magnetic field round the inductance, while in turn the E.M.F. set up by the changing magnetic field around the inductance assists in charging the condenser. The out-of-phase or reactive component of the E.M.F. has only to supply the difference between the vectors $\mathcal{V}_L$ and $\mathcal{V}_C$ and will be equal to $\omega L\,\mathcal{I} - \dfrac{\mathcal{I}}{\omega C}$. It may be noted that the second term of this voltage may in

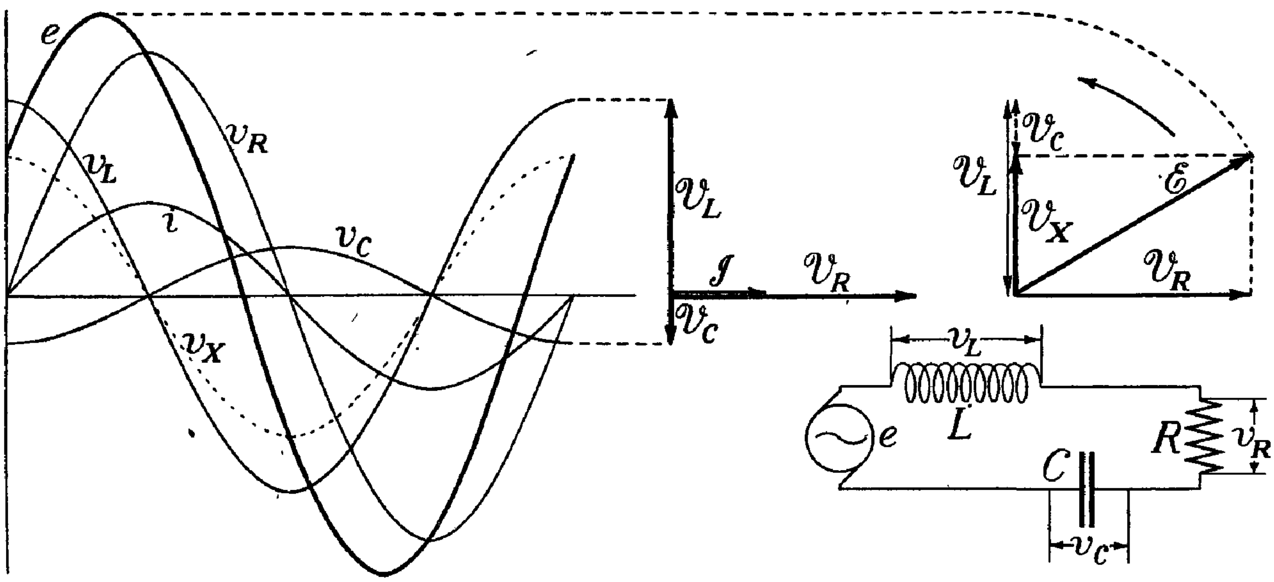


FIG. 25, CHAP. V.—Effect of resistance, inductance and capacitance in series.

some instances be larger than the first. If $\omega L\mathcal{I}$ is greater than $\dfrac{\mathcal{I}}{\omega C}$ the reactive voltage will be positive and will lead on the current, while if $\dfrac{\mathcal{I}}{\omega C}$ is greater than $\omega L\mathcal{I}$ the reactive voltage will be negative and will lag on the current. The effective reactive voltage being denoted by $\mathcal{V}_X$, the vector diagram shows that $\mathcal{E} = \sqrt{\mathcal{V}_R{}^2 + \mathcal{V}_X{}^2}$ that is, $\mathcal{E} = \mathcal{I}\sqrt{R^2 + \left(\omega L - \dfrac{1}{\omega C}\right)^2}$ or in R.M.S. values

$$I = \frac{E}{\sqrt{R^2 + \left(\omega L - \dfrac{1}{\omega C}\right)^2}}$$

the quantity $\sqrt{R^2 + \left(\omega L - \dfrac{1}{\omega C}\right)^2}$ is called the total impedance of the circuit.

Example 6.—A condenser of ·0015 microfarads, a resistance of 20 ohms and an inductance of 25 microhenries are connected in series, and an alternating E.M.F. of 1·5 millivolts R.M.S., at a frequency of 10^6 cycles per second, is applied to the circuit. Find (i) the R.M.S. value of the current (ii) the R.M.S. voltages across the condenser and inductance respectively, and (iii) the angle of phase difference.

It will be noted that the circuit is the same as in example 5 except that an inductance has been inserted in series.

$$\omega = 2\pi f = 2\pi \times 10^6$$

$$X_C = \frac{1}{\omega C} = 106 \text{ ohms}$$

$$X_L = \omega L = 2\pi \times 10^6 \times 25 \times 10^{-6} = 157 \text{ ohms}$$

$$X = \omega L - \frac{1}{\omega C} = 157 - 106 = 51 \text{ ohms}$$

$$Z = \sqrt{R^2 + X^2}$$
$$= \sqrt{20^2 + 51^2}$$
$$= \sqrt{3001}$$
$$= 54 \cdot 8 \text{ ohms.}$$

(i) $$I = \frac{E}{Z} = \frac{1 \cdot 5}{54 \cdot 8} \text{ milliamperes}$$
$$= \cdot 0274 \text{ milliamperes.}$$

In example 5 the R.M.S. current was only ·0139 milliampere. The introduction of the inductance has caused an increase of current amounting to about 100 per cent.

(ii) $$V_C = IX_C = \frac{I}{\omega C}$$
$$= \cdot 0274 \times 106 \text{ millivolts}$$
$$= 2 \cdot 9 \text{ millivolts}$$
$$V_L = IX_L = \omega L I$$
$$= \cdot 0274 \times 157 \text{ millivolts.}$$
$$= 4 \cdot 3 \text{ millivolts.}$$

Both V_L and V_C are greater than the R.M.S. applied voltage. The total reactive voltage is $V_L - V_C$ or 1·4 millivolts R.M.S., and this is equal to the reactive component of the alternator E.M.F.

(iii) The total reactance is $X_L - X_C = 51$ ohms. Since X_L is greater than X_C, the current will lag on the applied E.M.F. by an angle $tan^{-1}\frac{X}{R}$.

$$\frac{X}{R} = \frac{51}{20} = 2 \cdot 55$$
$$\theta = tan^{-1}\, 2 \cdot 55 = 69^\circ \text{ nearly.}$$

Effective inductance or capacitance

27. From the foregoing it will be observed that the reactive voltage $I\left(\omega L - \frac{1}{\omega C}\right)$ may be either leading or lagging on the current. Now inductive reactance causes the voltage to lead, and capacitive reactance causes it to lag. If $\omega L - \frac{1}{\omega C}$ is positive, it must have the same effect as an inductive reactance ωL_e, where L_e is the effective inductance of the components L and C in series, and

$$\omega L - \frac{1}{\omega C} = \omega L_e$$

$$\text{or } L_e = L - \frac{1}{\omega^2 C}$$

The effect of the capacitance is, in this particular instance, to reduce the apparent inductance L by an amount $\frac{1}{\omega^2 C}$. This is apparent from fig. 25, for the angle of phase difference is less with the condenser in circuit than if it were absent, and the impedance is also decreased with a proportional increase of current. If however the reactance $\omega L - \frac{1}{\omega C}$ is negative, the total effect of L and C must be that of a condenser, the equivalent value of which can now be found. If the latter is denoted by C_e,

$$\omega L - \frac{1}{\omega C} = - \frac{1}{\omega C_e}$$

$$\omega^2 L C - 1 = - \frac{C}{C_e}$$

$$\text{or } C_e = \frac{C}{1 - \omega^2 L C} .$$

Example 7.—(i) In example 5 what is the effective inductance of the circuit ?

The effective reactance has been found to be 51 ohms.

$$\omega L_e = 51$$

$$\omega = 2 \pi \times 10^6$$

$$\therefore L_e = \frac{51}{2\pi \times 10^6} \text{ henries}$$

$$= \frac{51}{2\pi} \text{ or } 8 \cdot 1 \text{ microhenries.}$$

(ii) If in example 6 the frequency is changed to $0 \cdot 6 \times 10^6$ what is the effective capacitance of the circuit ?

Instead of proceeding as above, the formula $C_e = \frac{C}{1 - \omega^2 L C}$ may be used

$$\omega = 2 \pi \times \cdot 6 \times 10^6$$

$$= 3 \cdot 77 \times 10^6$$

$$\omega L = 3 \cdot 77 \times 10^6 \times 25 \times 10^{-6}$$

$$= 94 \cdot 25$$

$$\omega C = 3 \cdot 77 \times 10^6 \times 0015 \times 10^{-6}$$

$$= \cdot 00565$$

$$\omega^2 L C = 94 \cdot 25 \times \cdot 00565$$

$$= \cdot 532$$

$$1 - \omega^2 L C = \cdot 468$$

$$\therefore C_e = \frac{C}{\cdot 468}$$

$$= \frac{\cdot 0015}{\cdot 468} \text{ microfarads}$$

$$= \cdot 0032 \text{ microfarads.}$$

It has now been shown that a circuit possessing both capacitance and inductance in addition to its resistance, may behave at certain frequencies as though it possessed no capacitance, but an amount of inductance smaller than that actually existent, while at other frequencies it may behave as though it possessed no inductance, but an amount of capacitance greater than the actual value.

POWER MEASUREMENT IN HEAVY-CURRENT PRACTICE

28. The amount of power which is dissipated in a direct current circuit is given by the equation $P = I^2 R$. In an A.C. circuit the power is given by an identical expression, provided that the R.M.S. value of the current is employed, because this value is by definition the direct current which is equivalent in heating effect to the given alternating current. In an A.C. circuit possessing resistance only, the power may also be calculated from the product of R.M.S. amperes and R.M.S. volts, because the R.M.S. voltage is defined by means similar to those adopted for the definition of R.M.S. current. In circuits possessing reactance, however, the product of volts and amperes does not give the true power expended, but a quantity called the activity, apparent power, or simply the volt-amperes. In a circuit possessing both capacitive and inductive reactance, as well as resistance, it has been stated that the applied E.M.F. consists of three components, namely (i) $v_R = i\,R$ which is required to overcome the resistance, and is in phase with the E.M.F. (ii) $v_L = \omega L\, i$ which overcomes the counter-E.M.F. of self-induction. This component may be considered to establish the magnetic field in and around the coils constituting the inductance. (iii) $v_C = \frac{i}{\omega C}$ which is devoted to the establishment of an electric field between the plates of the condenser.

No average power is supplied from the source of E.M.F. in order to maintain the magnetic and electric fields; the energy required to establish them is returned to the source on their destruction. The voltages v_L and v_C are 90° out of phase with the current, and are referred to as wattless components. The R.M.S. current is given by the equation $I = \frac{E}{Z}$ and the power expended in the circuit by $I^2 R$. Hence $P = \frac{E^2}{Z^2} R = E \frac{E}{Z} \frac{R}{Z} = E\, I \cos \varphi$, where φ is the phase difference and may be either a leading or lagging angle.

In any A.C. circuit a hot-wire or electrostatic voltmeter connected across the supply terminals will give the R.M.S. value of the terminal P.D., V, and a hot-wire ammeter in series with the consuming device will read the R.M.S. current, I. The product of these readings gives the apparent power, $V\,I$, and the true power is the product multiplied by $\cos \varphi$. The numeric $\cos \varphi = \frac{R}{Z}$ is therefore called the Power Factor of the circuit. True power may however be measured directly by means of a wattmeter, and the Power Factor may be determined by the relation

$$\cos \varphi = \frac{\text{True power}}{\text{apparent power}}$$

$$= \frac{\text{wattmeter reading}}{\text{Product of voltmeter and ammeter readings.}}$$

The wattmeter

29. Two principal types of wattmeter are in use, and are known as the dynamometer and induction types respectively. Hot-wire and electrostatic types have also been proposed but have not been developed into practical instruments. The dynamometer instrument is similar in principle to the moving coil D.C. instrument, but contains no permanent magnet. It is shewn diagrammatically in fig. 26. An alternating magnetic field is set up by a fixed coil carrying the main current or a definite fraction thereof, and the strength of this field, at any instant, is proportional to the instantaneous current. The moving coil is situated in this field, and is connected across the supply mains with a suitable resistance in series. The current in this coil is therefore proportional to the terminal P.D. Thus the connections of the fixed coil resemble those of an ammeter and the connections of the moving coil those of a voltmeter. No iron is used in the vicinity of the coils.

The torque exerted upon the moving coil is proportional to the product of the two currents and therefore proportional to the product of the main current and terminal P.D., that is to the

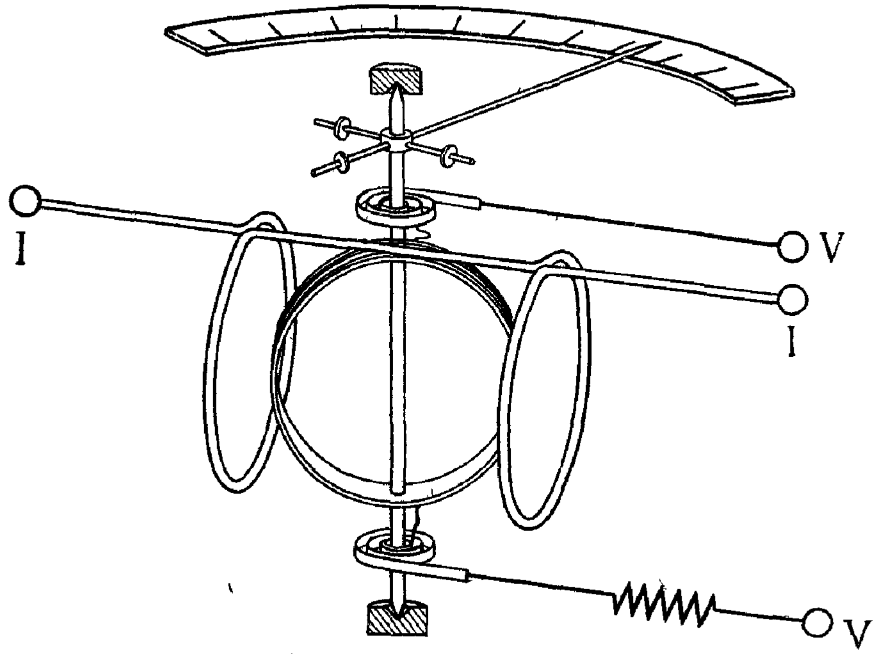


FIG. 26, CHAP. V.—Principle of dynamometer instrument.

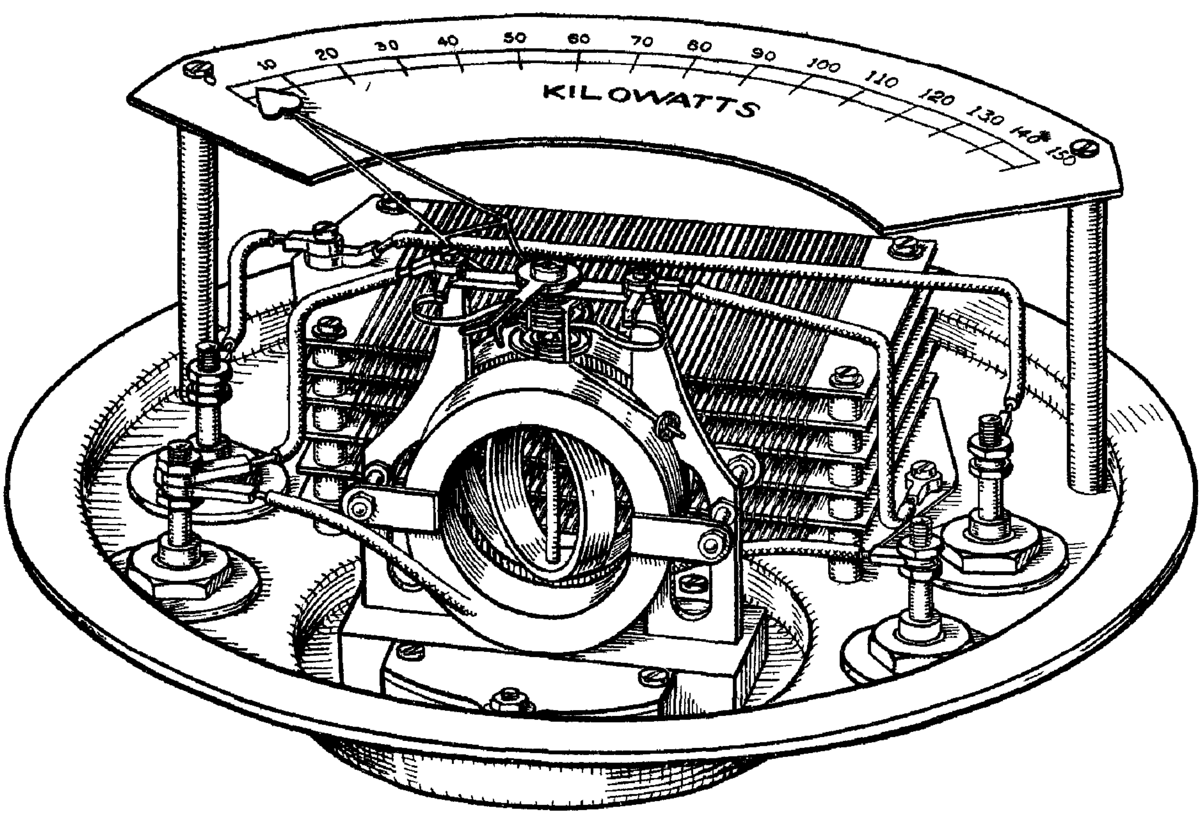


FIG 27, Chap. V.—Dynamometer wattmeter.

instantaneous power. The moving system cannot follow the rapid changes in the latter quantity, however, but takes up some position in which the controlling torque is equal to the mean value of the deflecting torque, that is the mean power, VI.

The controlling torque is obtained by spiral springs which also serve as connecting leads to the moving coil, and the movement is made deadbeat by an air damping device. The general appearance of a typical wattmeter of this type is shown in fig. 27.

The induction wattmeter

30. In this type of instrument the moving member consists of a thin aluminium disc, which is mounted upon a spindle, and is free to rotate through about 300° against the action of a light spiral spring. An electromagnet is arranged on each side of the disc, the poles of each being opposite to a different portion. The winding of one electromagnet carries the main current or a known fraction thereof, while the other winding is connected across the mains and carries a current which is proportional to the P.D. The latter winding has a series inductive resistance which causes its current to lag by nearly 90° upon the terminal P.D. The alternating fluxes set

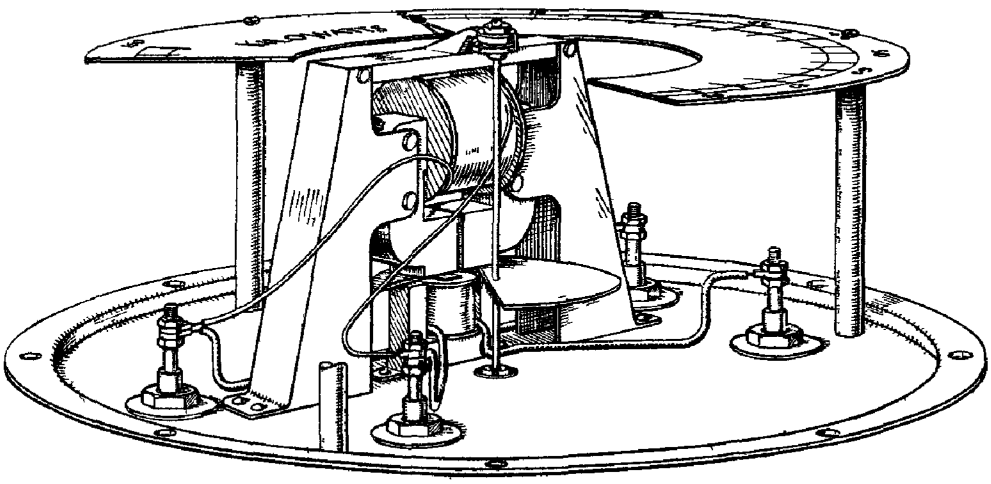


FIG. 28, CHAP. V.—Induction wattmeter.

up by these two magnets induce local electromotive forces in the disc, and consequently eddy currents are set up in the latter. These eddy currents in turn set up a flux which reacts upon the original flux, and a torque is exerted upon the disc which is proportional to the product of the fluxes caused by the two electromagnets and also to the sine of the engle of phase difference between them. Thus the torque is proportional to $VI \sin (90° - \varphi)$ or $VI \cos \varphi$. A diagrammatic representation of this instrument is given in fig. 28. The instrument requires certain compensating devices (not shown on this figure) in order to avoid inaccuracy in indication, because it is impossible to make the parallel winding so highly inductive (or slightly resistive) that the current in it will lag by 90° on the terminal P.D.

Power measurement by voltmeter

31. When it is necessary to measure power or power factor in an A.C. circuit and a wattmeter is not fitted, the same information can be obtained in the following manner.

In fig. 29 (S_1) (S_2) are the supply terminals, and Z is the device whose power or power factor are to be measured. In series with Z is connected a noninductive resistance R such as a number of carbon filament lamps or an electric radiator; a hot-wire voltmeter (including of course its series resistance) is arranged to read at will either V_1, the P.D. between the terminals of the device Z; V_2, the P.D. between the terminal of the noninductive resistance R, or V_3 the P.D. between the terminals of R and Z in series. An ammeter is connected in the supply line.

The relation between the voltages V_1, V_2 and V_3 can be shewn by a vector diagram, which has been drawn beside the circuit diagram for easy reference. The vector I represents the current; the P.D. V_2 across the resistance is in phase with I, while the voltage V_1 across Z leads upon I by some angle φ. The supply voltage V_3 is the vector sum of V_1 and V_2. From the end of the vector V_3 a perpendicular to the current vector is drawn, this being shown as a dotted line in the figure. This completes a right-angled triangle the hypoteneuse of which is V_3. By inspection we find that the two other sides of this triangle are ($V_1 \cos\varphi + V_2$) and ($V_1 \sin\varphi$). From this information we deduce that

$$V_3^2 = (V_1 \cos\varphi + V_2)^2 + (V_1 \sin\varphi)^2$$

$$V_3^2 = V_1^2 \cos\varphi + 2\, V_1\, V_2 \cos\varphi + V_2^2 + V_1^2 \sin^2\varphi$$

But $$\sin^2\varphi + \cos^2\varphi = 1$$

Hence $$V_3^2 = V_1^2 + V_2^2 + 2V_1\, V_2 \cos\varphi$$

and $$V_1 \cos\varphi = \frac{V_3^2 - V_1^2 - V_2^2}{2V_2}$$

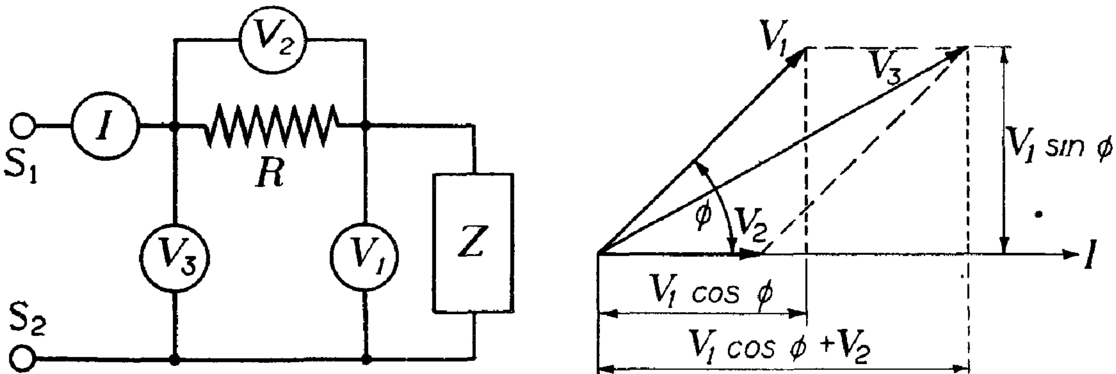


FIG. 29, CHAP. V.—Power measurement in A.C. circuit (Three voltmeter method).

The current flowing, as shown by the ammeter, can now be introduced giving

$$V_1 I \cos\varphi = \frac{V_3^2 - V_1^2 - V_2^2}{2V_2}\, I$$

which is the power consumed by the device Z. As the power is deduced from the differences between the squares of quantities, small errors in these quantities, i.e. in the voltmeter readings, produce considerably greater errors in the value of the power. For best results the series resistance should be so chosen that its value is about equal to the impedance of the device under measurement. This necessitates a supply voltage at least 50 per cent. higher than that for which Z is rated, while in the unlikely event of the latter being completely non-reactive, the normal supply voltage would have to be doubled for the purposes of the measurement. This disadvantage can be overcome by using the following method of measurement.

Power measurement by ammeter

32. The general principles of this method are similar to those in the preceeding. The connections are shown in fig. 30 in which the device under measurement is again Z. Ammeters are connected in each branch of the circuit, the noninductive resistance R with its meter being placed in parallel with Z. From the vector diagram it is deduced in exactly the same manner as before that

$$V I_1 \cos\varphi = \frac{I_3^2 - I_2^2 - I_1^2}{2I_2}\, V$$

The disadvantage of this method is that three ammeters of identical accuracy are required, otherwise elaborate switching arrangements are necessary in order to transfer the ammeter from circuit to circuit. Both methods assume that the wave form is sinusoidal and inaccurate results are obtained if this assumption is incorrect.

33. Power factor meters may be used to indicate directly the power factor of a circuit. A typical design consists of a fixed coil which carries the main current, this coil being divided into two halves. Pivoted in the space between these is the moving element consisting of two coils rigidly fixed at right-angles to each other so that they move as one unit. No controlling torque is required. One coil of the moving element in series with a noninductive resistance is connected across the supply mains, the winding of the other being similarly connected but with a reactive device, i.e. a condenser or inductance, in series, and consequently the former winding carries a current which is proportional to and in phase with the voltage, while the latter winding carries a current which is proportional to the voltage but which differs considerably in phase. It will be considered as a lagging current in the following explanation.

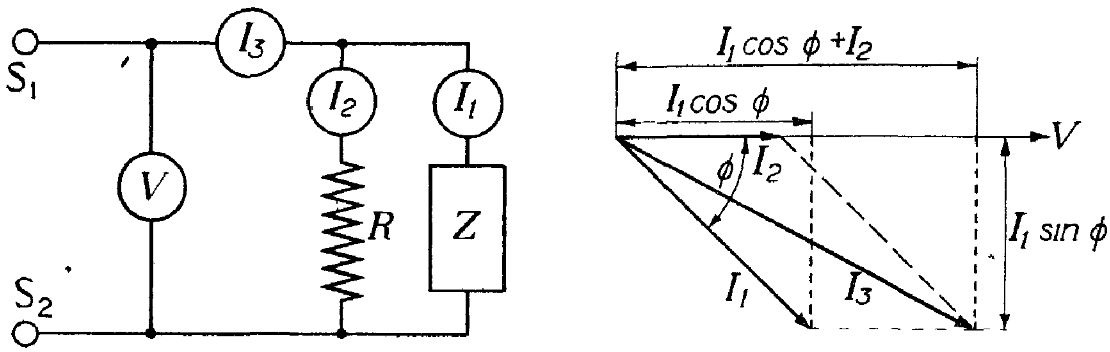


FIG. 30, CHAP. V.—Power measurement in A.C. circuit (Three ammeter method).

Suppose the main circuit to have unity power factor, then the current in the non-inductive branch of the moving unit will interact with the main field in such a way that its coil will set itself parallel to the main coil so that it embraces maximum flux; on the other hand if the lag of current in the main coil is exactly equal to the lag of current in the inductive branch of the moving unit, the latter will be turned until the coil embraces maximum flux, and will be parallel to the main coil. For any intermediate angles of lag the moving unit takes up an intermediate position in which the resultant field of the moving unit is parallel to the main field at the instant of peak value of the latter. With a leading current in the main coil, the same arguments apply, except that the moving coil in the inductive branch will turn in the opposite direction to that in which it turned with a lagging main current. The pointer attached to the moving element moves over a scale graduated up to 90° on either side of zero, thus showing the phase angle and whether the current is leading or lagging.

Energy meters

34. Energy meters for A.C. supply are usually true watt-hour meters. The typical form is similar in principle to the induction type of wattmeter. The chief modification is the removal of the spiral spring constituting the controlling force of the wattmeter, so that the spindle carrying the disc is free to rotate in its bearings, and the provision of a counting mechanism, e.g. a cyclometer, instead of a pointer. This cyclometer is driven from the spindle of the disc by a worm cut in the latter.

A retarding torque is provided by an additional permanent magnet, by which eddy currents are induced in the revolving disc, and consequently this torque is directly proportional to the speed, while the torque exerted by the electromagnets is proportional to the power supplied or to $VI \cos \varphi$. Hence the total number of revolutions is proportional to the watt-hours.

Rating of alternating current machinery

35. Makers of alternating current machinery rate their products as being capable of delivering a given number of kilo-volt-amperes instead of a given number of kilo-watts, e.g. an alternator may be spoken of as a 200 volt, 10 kVA machine. This means that at its rated speed it will deliver the rated voltage and is capable of delivering 10,000 volt-amperes without overheating. If the load is purely inductive, this current will be wattless, and no power will be dissipated in the external circuit, although the machine is giving an output of 10 kVA and the internal losses are exactly the same as when current is delivered to a power-dissipating circuit. On the other hand if the maker guaranteed his alternator to produce 10 kW at 200 volts irrespective of the nature of the load, and the machine were called upon to deliver this power to a load having a power factor of $0{\cdot}5$, the apparent power would be $\frac{10{,}000}{0{\cdot}5} = 20{,}000$ volt-amperes, and the current would be $\frac{20{,}000}{200} = 100$ amperes, or double the current required by a load possessing a power factor of unity. With the reactive load the heating effect in the machine itself is obviously four times that caused by the non-reactive load, and the machine would certainly suffer damage.

SERIES RESONANCE

36. We have seen that under certain conditions the inductive and capacitive effects in a circuit tend to cancel each other. This cancellation is complete if the values of L, C and ω are such that $\omega L - \frac{1}{\omega C} = 0$. The counter-E.M.F. of self-induction, ωLi, and the P.D. at the condenser terminals, $\frac{i}{\omega C}$, are then equal in magnitude and opposite in phase at every instant throughout the cycle, and the circuit will behave as if it had neither inductance nor capacitance. The circuit is then said to be in series resonance with the frequency of the applied E.M.F.

The resonant frequency of a circuit possessing capacitance and inductance in series may be defined as the frequency at which the total reactance is zero. For given values of L and C the resonant frequency f_r is found by equating ωL to $\frac{1}{\omega C}$ thus:—

$$\omega L = \frac{1}{\omega C}$$

$$\omega^2 = \frac{1}{LC}$$

$$\omega = \frac{1}{\sqrt{LC}} = 2\pi f_r \quad \therefore \quad f_r = \frac{1}{2\pi\sqrt{LC}}$$

As the reactance of a circuit to an E.M.F. at its resonant frequency is zero, the current at this frequency must depend only upon the resistance of the circuit, and is given by the equation

$$i = \frac{\mathscr{E} \sin \omega t}{R}$$

the R.M.S. value being $$I = \frac{E}{R}$$

This gives an alternative definition of series resonance, i.e. the frequency at which the current has the value $\frac{E}{R}$. In a resonant circuit, the current and E.M.F. are in phase, and the power factor is unity.

The term resonance is borrowed from the science of acoustics, and numerous examples occur in all branches of physics, many being matters of everyday experience. For instance, a wine glass which emits a clear note of definite pitch when tapped sharply, will emit a similar sound if the appropriate note is produced by a musical instrument in the vicinity. The action of the reed type frequency meter (para. 7) depends upon this principle, the natural frequency of vibration of one particular reed being coincident with the frequency of the current flowing through the magnet winding.

Electrical resonance is not often met with in heavy current engineering, and when a possibility of resonance exists it is generally suppressed by suitable variation of the circuit constants. It will be shown that under resonant conditions the voltage across certain circuit components may be many times the applied voltage, and in power circuits this increase of voltage would necessitate very heavy insulation, besides leading to other complications into which it is unnecessary to enter. In radio circuits, however, and particularly in receivers where the applied voltage is often only of the order of a few microvolts, the phenomenon of resonance is utilised in order to achieve effects greater than could be obtained by direct employment of the available voltage. For this reason further discussion of resonance will be illustrated by examples of direct application to radio-communication. The terms " audio-frequency " and " radio-frequency " have already been introduced, the range included in the latter term being from 20,000 cycles per second to several million cycles per second. It is convenient to refer to frequencies of this order in kilocycles per second (kc/s) or megacycles per second (Mc/s), while it has also been proposed to use the term hertz to denote one cycle per second, but this unit has not yet been adopted for service use. In radio-frequency circuits the inductance and capacitance are invariably only small fractions of a henry and farad respectively, and the units used are the microhenry and microfarad.

It has been shown that the resonant frequency of a circuit possessing an inductance of L henries and a capacitance of C farads is given by the formula

$$f_r = \frac{1}{2\pi\sqrt{LC}}$$

It is often more convenient to use a formula giving the resonant frequency in terms of the inductance in microhenries, and the capacitance in microfarads. This is derived as follows :—

Let L = the inductance of the circuit, in microhenries.

C = the capacitance of the circuit, in microfarads

f_r = the resonant frequency of the circuit.

Since 1 henry = 10^6 microhenries

1 farad = 10^6 microfarads

$$f_r = \frac{1}{2\pi\sqrt{\frac{L}{10^6} \times \frac{C}{10^6}}}$$

$$= \frac{10^6}{2\pi\sqrt{LC}}.$$

When this relation is satisfied, the series circuit is said to be an accoptor circuit for the frequency f_r.

Series resonance curves

37. Referring to the circuit shown in fig. 25, let $L = 150\ \mu H$, $C = 000169\ \mu F$, $R = 10$ ohms and the E.M.F. of the alternator to be 10 millivolts (R.M.S.). Suppose the frequency to be variable between say 950 kc/s and 1,050 kc/s. The current at any frequency $f = \frac{\omega}{2\pi}$ is given by the equation

$$I = \frac{E}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}},$$

bearing in mind that L denotes the inductance in henries and C the capacitance in farads. As the frequency is varied between the given limits, the current will also vary; its value has been calculated over the range 970 to 1,030 kc/s, and the results plotted in fig. 31 curve (i). It will be seen that the current reaches the value $\frac{E}{R}$, i.e. one milliampere, when the supply frequency is 1,000 kc/s, which is the resonant frequency. On either side of resonance, the current is less than this, falling off rapidly at first and then more slowly. At frequencies below 1,000 kc/s the, capacitive reactance $\frac{1}{\omega C}$ is greater than the inductive reactance ωL and the current leads upon

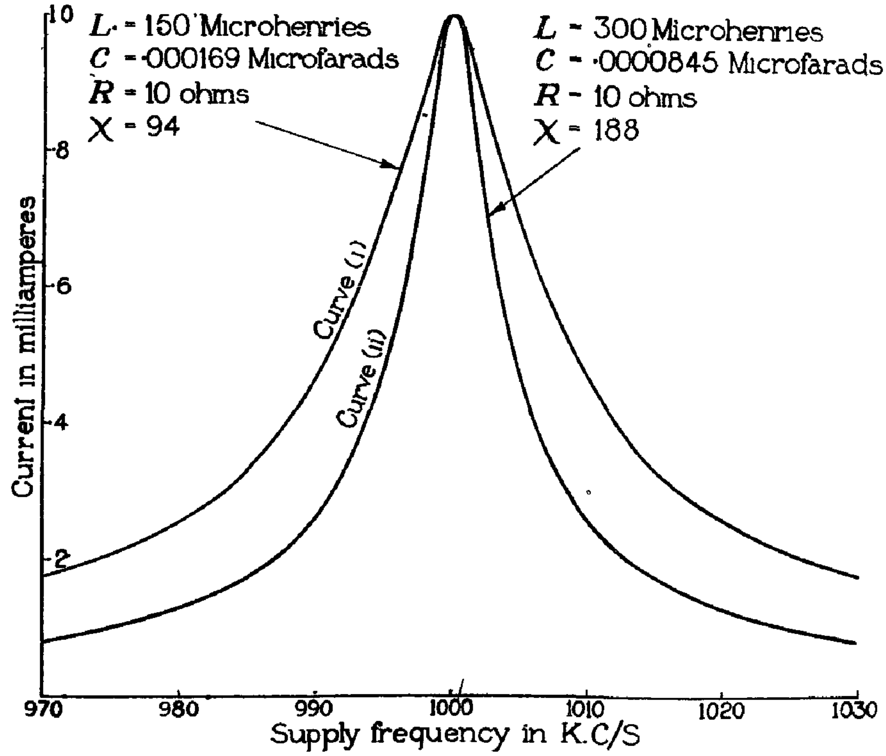


FIG. 31, CHAP. V.—Series resonance curves. Effect of ratio of inductance to capacitance.

the applied voltage, while at frequencies above 1,000 kc/s the inductive reactance is greater than the capacitive reactance and the current lags on the applied voltage. The graph showing the variation of current as the frequency is varied is called the resonance curve of the circuit.

Selectivity of an acceptor circuit

38. One of the principal applications of the phenomenon of electrical resonance is in the radio receiving circuit. A distant radio transmitter sets up in a receiving aerial an alternating E.M.F., the frequency of which is the same as that of the transmitter. More complete consideration of both transmitters and receivers will be found in subsequent chapters, but for the present it may be considered that all transmitters of equal power, situated at the same distance from the receiver, may be expected to produce equal E.M.F.'s in the receiving aerial. (Certain qualifications of this assumption are necessary and will be found in the appropriate chapters.) The receiving aerial circuit possesses inductance, capacitance, and resistance and may be represented as in fig. 32, where the alternators E_1, E_2, E_3 represent induced E.M.F.'s, having frequencies f_1, f_2, f_3 respectively. These alternators therefore give the same effect in the circuit as three different transmitters, and if $f_1 = 990$ kc/s, $f_2 = 100$ kc/s and $f_3 = 1{,}020$ kc/s, fig. 31 curve (i)

shews that if E_1, E_2 and E_3 are each equal to 10 millivolts, E_1 will produce a current of ·46 milliampere, E_2 a current of 1 milliampere and E_3 a current of ·26 milliampere. Although the three voltages applied to the circuit are of equal value, the one which has a frequency equal to the resonant frequency of the circuit will produce the largest current, and therefore the strongest signal in the telephone receivers or loud speaker of the receiver. The inclusion of an acceptor

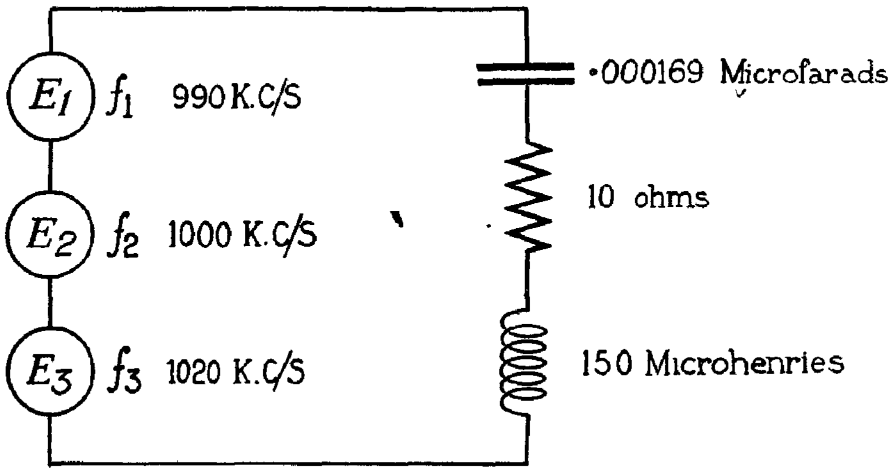


FIG. 32, CHAP. V.—Series L/C circuit with applied E.M.F. at resonant and non-resonant frequencies

circuit in a piece of apparatus thus endows it with the property of discrimination in favour of signals (i.e. E.M.F.'s) of resonant frequency, to the partial exclusion of others. This capability to differentiate between signals of different frequencies is called selectivity, and the selectivity of an acceptor circuit depends upon its ratio of inductance to resistance.

Influence of ratio $\frac{L}{R}$

39. The effect of an increase in the ratio $\frac{L}{R}$ is also shown in fig. 31. Curve (i) has already been referred to; it is the resonance curve for a ratio $\frac{L}{R}$ of $\frac{150}{10} = 15$. In curve (ii) the inductance has been increased to 300 μH, the capacitance being correspondingly reduced so that the resonant frequency remains 1,000 kc/s, the ratio $\frac{L}{R}$ being 30. The three E.M.F.'s cited above would now produce the following currents, viz: at 990 kc/s, ·23 milliampere; at 1,000 kc/s 1 milliampere, as before, and at 1,020 kc/s ·13 milliampere. With this ratio of $\frac{L}{R}$, then, the current at resonance is unchanged, but E.M.F.'s of non-resonant frequencies cause considerably reduced currents to flow, and the selectivity is therefore increased.

The ratio $\frac{L}{R}$ may also be changed by a variation of resistance, instead of by variation of inductance. Fig. 33 shows resonance curves for fixed values of inductance and capacitance (150 μH and ·000169 μF respectively); curve (i) representing the state of affairs when the resistance is 10 ohms, is repeated from fig. 31 for comparison. Curve (ii) shows the effect of increasing the resistance to 14·14 ohms and curve (iii) the effect of decreasing the resistance to 7·07 ohms. Comparing the two latter curves, it is seen that at the resonant frequency, halving the resistance results in doubling the current, but at any other frequency the effect is not so marked. At 970 kc/s and at 1,030 kc/s the current is practically independent of the resistance and depends only upon the reactance of the circuit, so that the current is the same in all three cases.

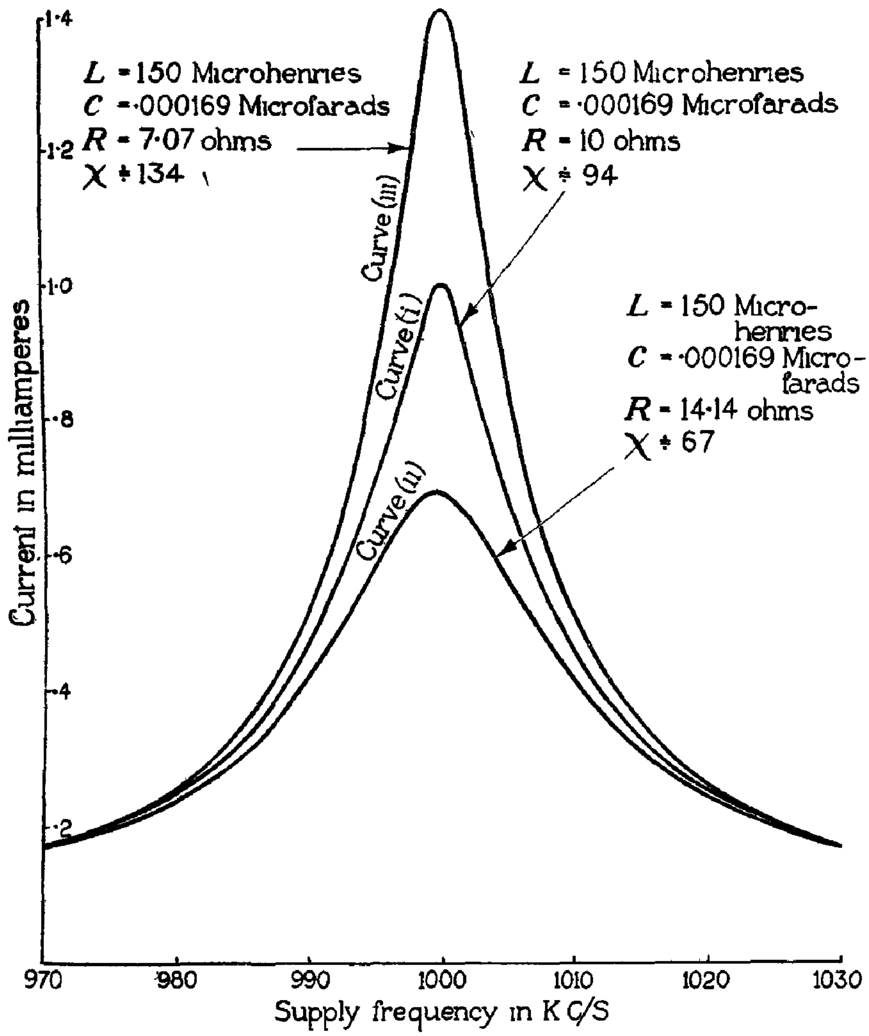


FIG. 33, CHAP. V.—Series resonance curves. Effect of variation of resistance.

40. The reader must be on his guard against a common fallacy, i.e. that an increase in the ratio $\frac{L}{C}$ will necessarily lead to an increase in the selectivity of an acceptor circuit. This is not true unless the increase of inductance is accompanied by an increase in the ratio $\frac{L}{R}$ which in itself is sufficient to cause an increase of selectivity. As an example, consider the selectivity of two different acceptor circuits. Let R_1, L_1, C_1 be the constants of one circuit, and R_2, L_2, C_2 those of the other, also let $R_2 = 2R_1$, $L_2 = 2L_1$, $C_2 = \frac{1}{2}C_1$; the circuits then have the same resonant frequency and the same ratio of inductance to resistance, but the ratio of inductance to capacitance in the circuit $R_2\,L_2\,C_2$ is four times as great as in the circuit $R_1\,L_1\,C_1$. Let I_r denote the current at the resonant frequency $\frac{\omega_r}{2\pi}$, and I_n the current at any other frequency, $\frac{\omega_n}{2\pi}$.

In the first circuit

$$I_r = \frac{E}{R_1}, \; I_n = \frac{E}{\sqrt{R_1^2 + X_1^2}} = \frac{E}{Z_1},$$

where $X_1 = \omega_n L_1 - \dfrac{1}{\omega_n C_1}$.

In the second circuit

$$I_r = \frac{E}{R_2}, \; I_n = \frac{E}{\sqrt{R_2^2 + X_2^2}}\;; \; X_2 = \omega_n L_2 - \frac{1}{\omega_n C_2}.$$

$$X_2 = 2\omega_n L_1 - \frac{2}{\omega_n C_2} = 2X_1\;; \quad \therefore \; I_n = \frac{E}{\sqrt{(2R_1)^2 + (2X_1^2)}} = \frac{E}{2Z_1}.$$

For the present purpose the selectivity may be defined as the ratio of current at resonance to current at the non-resonant frequency, that is $\dfrac{I_r}{I_n}$.

In the first circuit

$$\frac{I_r}{I_n} = \frac{E}{R_1} \div \frac{E}{Z_1} = \frac{\sqrt{R_1^2 + X_1^2}}{R_1}.$$

In the second circuit

$$\frac{I_r}{I_n} = \frac{E}{2R_1} \div \frac{E}{2Z_1} = \frac{\sqrt{R_1^2 + X_1^2}}{R_1};$$

as I_n is the current at any non-resonant frequency whatever, it is seen that although the maximum value of the current is different, the two resonance curves will have exactly the same shape, and the two circuits have the same selectivity, because the ratio $\dfrac{I_r}{I_n}$ is the same in each example.

Voltage magnification

41. It has already been shown that in a circuit possessing both capacitance and inductance connected in series, the terminal P.D. of the coil or condenser may be greater than the applied E.M.F. This effect is most pronounced when the frequency of the applied E.M.F. is identical with the resonant frequency of the circuit. Let E be the applied E.M.F. (R.M.S.) L the value of the inductance in henries, C the capacitance of the condenser in farads, and R the resistance of the circuit in ohms. If V_L is the P.D. at the terminals of the coil, V_C the P.D. at the condenser terminals, and $\omega_r = 2\pi$ times the resonant frequency,

$$V_L = \omega_r L I$$

$$V_C = \frac{1}{\omega_r C} I$$

Since at resonance, $I = \dfrac{E}{R}$ and $\omega_r L = \dfrac{1}{\omega_r C}$,

$$V_L = V_C.$$

$$V_L^2 = V_L V_C = \frac{\omega_r L E}{R} \times \frac{E}{\omega_r C R}$$

$$= \frac{E^2 L}{R^2 C}$$

$$\therefore V_L = V_C = \frac{E}{R}\sqrt{\frac{L}{C}}$$

Example 8.—A coil having an inductance of 160 microhenries, a condenser having a capacitance of ·00025 microfarads, and a resistance of 10 ohms are placed in series with an E.M.F. of 2 volts (R.M.S.) at the resonant frequency. Find the R.M.S. voltage at the terminals of the coil and condenser.

$$V_L = V_C = \frac{E}{R}\sqrt{\frac{L}{C}}$$
$$= \frac{2}{10}\sqrt{\frac{160 \times 10^{-6}}{\cdot 00025 \times 10^{-6}}}$$
$$= \cdot 2\sqrt{640000}$$
$$= 160 \text{ volts.}$$

The ratio $\frac{V_L}{E}$ is called the resonance voltage magnification of the circuit, or if there is no danger of ambiguity, the circuit magnification. It may be denoted by the symbol χ. The appropriate values of this constant have been inserted on the curves in figs. 31 and 33. It will be seen that a circuit having a high value of χ gives a resonance curve which rises sharply as the resonant frequency is approached while with low values of χ the resonance curve tends to become flat. From this graphic point of view it has become the practice to speak of the sharpness of resonance; a circuit having high χ is said to be sharply resonant, and a circuit having low χ to be flatly resonant, or flatly tuned.

42. In practice the resistance of a radio-frequency circuit is generally an undesirable feature. In this respect wireless circuits differ from many alternating power circuits in which the desired effect is often the production of heat, sometimes for its own sake, as in electric radiators and soldering irons, and sometimes because the heat is required to render a body incandescent as in the electric lamp. The resistance of a radio circuit is often only that inherent in the coils and condensers, and the efficiency of the coil or condenser is given by the ratio of its reactance to its resistance. This ratio is often spoken of as the Q of the component, but as the symbol Q is used to denote quantity of electricity, the term figure of merit will be used to denote this ratio. The figure of merit of an inductive coil is therefore equal to $\frac{\omega L}{R}$ where $\frac{\omega}{2\pi}$ is the frequency, L the inductance of the coil in henries and R its resistance in ohms. This figure is approximately constant over a wide frequency range owing to the fact that the h.f. resistance of a coil is roughly proportional to the frequency. Thus if R_1 is the resistance of the coil at a frequency $f_1 = \frac{\omega_1}{2\pi}$ and R_2 the resistance at a frequency $f_2 = \frac{\omega_2}{2\pi}$, $R_2 \doteqdot \frac{\omega_2}{\omega_1} R_1$. At the frequency f_1 the figure of merit is $\frac{\omega_1 L}{R_1}$ while at the frequency f_2 it is $\frac{\omega_2 L}{R_2}$ and $\frac{\omega_2 L}{R_2} \doteqdot \frac{\omega_2 L}{\frac{\omega_2}{\omega_1} R_1} \doteqdot \frac{\omega_1 L}{R_1}$.

The closeness of this approximation may be illustrated by the measured values for a certain coil, which possessed an inductance of 185 microhenries. At a frequency of 500 kc/s the figure of merit was 120, rising to 160 over the range 800 to 1,000 kc/s. At higher frequencies the figure of merit decreased slowly, being again 120 at 1,500 kc/s.

If an inductance is connected in series with a loss-free condenser, and an E.M.F. applied at the resonant frequency of the circuit, the resonant voltage magnification is equal to $\frac{\omega_r L}{R}$ i.e. to the figure of merit of the coil at this particular frequency. For this reason the term "coil magnification" is sometimes used instead of figure of merit, although this may lead to confusion. The symbol χ_L may be used to denote the figure of merit of a coil. The efficiency of a condenser may also be expressed as a figure of merit which is the ratio of its reactance to its effective resistance, including in the latter all sources of energy loss. It may be denoted by the symbol χ_C. As the energy losses are approximately in inverse proportion to the frequency, χ_C is again fairly constant over a wide frequency range.

PARALLEL COMBINATIONS OF INDUCTANCE, CAPACITANCE AND RESISTANCE

43. In drawing the vector diagrams to show the relative magnitudes and phase difference of current and voltage in a series circuit, the current vector is used as a datum line, because the same current flows through every component. The resistive P.D. $R\,\mathscr{I}$ is drawn parallel to the vector $\mathscr{I}$ and the reactive P.D.'s perpendicular to the current vector, lagging or leading as the case may be. In parallel circuits the circuit components have a common terminal voltage and this vector is taken as the datum line; a vector representing the current through a resistance is drawn parallel to the voltage vector, and currents through purely reactive components perpendicular to the voltage vector, lagging or leading as requisite. The rule in drawing a vector diagram is therefore to use as a reference vector the one representing the quantity which is common to all circuit components. In the following paragraphs the discussion will be illustrated only with vector diagrams, although of course the corresponding sine and cosine curves could be used as in the case of series circuits.

Resistance and inductance in parallel

44. In fig. 34a is shown an inductance L and a resistance R connected in parallel, the inherent resistance of the inductance being negligible. An alternating E.M.F. of peak value $\mathscr{E}$ is

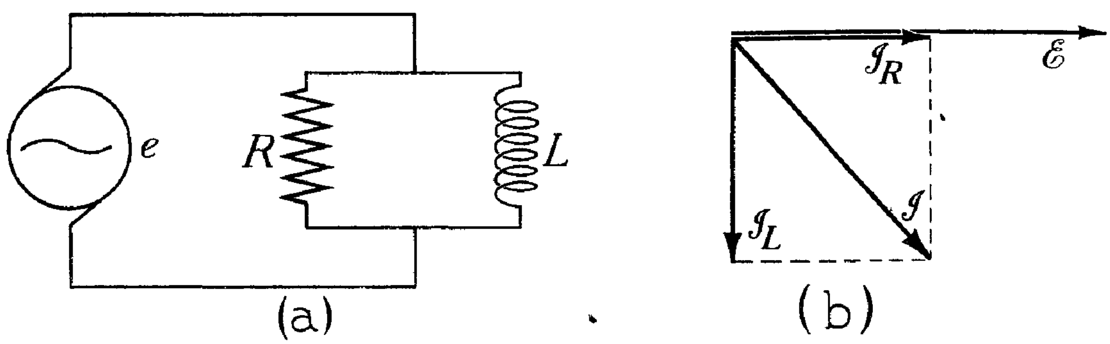


FIG. 34, CHAP. V.—Effect of resistance and inductance in parallel.

applied to the terminals of the combination, and consequently alternating currents of the same frequency will flow in both branches of the circuit. The current through the inductance will have a peak value $\mathscr{I}_L = \dfrac{\mathscr{E}}{\omega L}$ lagging on the applied voltage by 90°, while the current through the resistance will have the peak value $\mathscr{I}_R = \dfrac{\mathscr{E}}{R}$, and will be in phase with the applied voltage. The total current $\mathscr{I}$ supplied by the alternator will be the vector sum of $\mathscr{I}_L$ and $\mathscr{I}_R$, and is shown in the vector diagram fig. 34b, from which it is deduced that

$$\begin{aligned}\mathscr{I} &= \sqrt{\mathscr{I}_R^2 + \mathscr{I}_L^2}\\ &= \sqrt{\left(\frac{\mathscr{E}}{R}\right)^2 + \left(\frac{\mathscr{E}}{\omega L}\right)^2}\\ &= \mathscr{E}\sqrt{\left(\frac{1}{R}\right)^2 + \left(\frac{1}{\omega L}\right)^2}\end{aligned}$$

and this current will lag on the applied voltage by an angle

$$\theta = tan^{-1}\frac{\mathscr{I}_L}{\mathscr{I}_R}, \text{ i.e.}$$

$$tan\ \theta = \frac{\mathscr{I}_L}{\mathscr{I}_R} = \frac{\mathscr{E}}{\omega L} \times \frac{R}{\mathscr{E}} = \frac{R}{\omega L}$$

In R.M.S. values,
$$I = E\sqrt{\left(\frac{1}{R}\right)^2 + \left(\frac{1}{\omega L}\right)^2}$$

and the ratio $\frac{E}{I}$ is the impedance of the parallel combination. Hence

$$Z = \frac{E}{I} = \frac{1}{\sqrt{\left(\frac{1}{R}\right)^2 + \left(\frac{1}{\omega L}\right)^2}}$$

The reciprocal of the impedance is called the admittance and is denoted by Y. The reciprocal of the resistance $\frac{1}{R}$, is termed the conductance and is denoted by G, while the reciprocal of the inductive reactance $\frac{1}{\omega L}$ is termed the inductive susceptance, and is denoted by B_L. (The symbol B is also used for flux density, but as flux density and susceptance rarely occur in the same calculation there is little risk of confusion.) The relation between E and I may therefore be written

$$I = YE = \sqrt{G^2 + B_L^2}\ E$$

Resistance and capacitance in parallel

45. For the inductance L in the preceeding discussion let a capacitance C be substituted, and an E.M.F. of peak value $\mathscr{E}$ be applied to the parallel combination, fig. 35a. As before, an

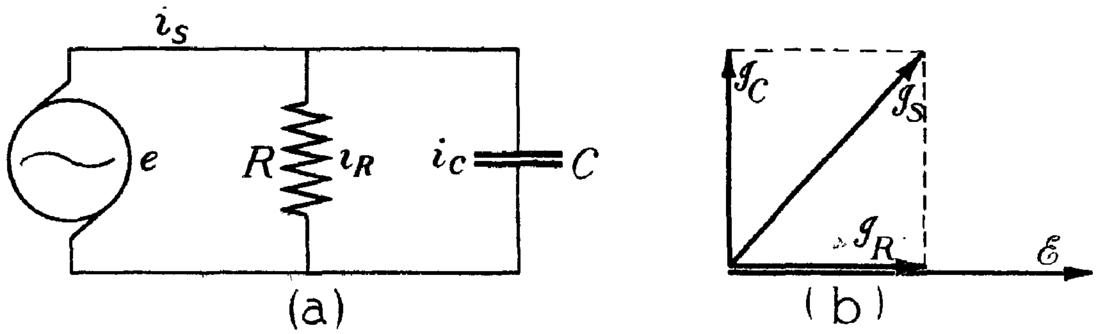


FIG. 35, CHAP. V.—Effect of resistance and capacitance in parallel.

alternating current of the supply frequency will flow in each component, the peak current through the resistance being $\mathscr{I}_R = \frac{\mathscr{E}}{R}$ in phase with the applied E.M.F. as before. The current charging the condenser will be $\mathscr{I}_C = \omega C\mathscr{E}$ and will lead on the applied E.M.F. by 90°. The total current $\mathscr{I}$ supplied by the alternator will be the vector sum of $\mathscr{I}_C$ and $\mathscr{I}_R$, and is shown in the vector diagram fig. 35b. It will be seen that

$$\mathscr{I} = \sqrt{\left(\frac{\mathscr{E}}{R}\right)^2 + (\omega C\mathscr{E})^2}$$

$$= \mathscr{E}\sqrt{\left(\frac{1}{R}\right)^2 + (\omega C)^2}$$

$$= \mathscr{E}\sqrt{G^2 + B_C^2}$$

and this current will lead on the applied voltage by an angle $\theta = tan^{-1} \dfrac{\mathcal{I}_C}{\mathcal{I}_R} = \omega CR$. The expression $\sqrt{\left(\dfrac{1}{R}\right)^2 + (\omega C)^2}$ is the admittance of the parallel combination, the capacitive susceptance being $B_C = \omega C$.

Example 9.—An inductance of 100 microhenries, having negligible resistance, and a resistance of 600 ohms, are placed in parallel. Find the R.M.S. current when an R.M.S. voltage of 2 volts is applied, at a frequency of 800 kc/s.

$$\omega = 2\pi f = 6{\cdot}28 \times 800{,}000$$
$$\omega L = 6{\cdot}28 \times 800{,}000 \times 100 \times 10^{-6} \text{ ohms}$$
$$R = 600 \text{ ohms}$$
$$I_L = \frac{E}{\omega L} = \frac{2}{502{\cdot}4} = {\cdot}00398 \text{ amperes or } 3{\cdot}98 \text{ milliamperes.}$$
$$I_R = \frac{E}{R} = \frac{2}{600} = {\cdot}00333 \text{ amperes or } 3{\cdot}33 \text{ milliamperes.}$$
$$I = \sqrt{I_R^2 + I_L^2}$$
$$= \sqrt{3{\cdot}33^2 + 3{\cdot}98^2}$$
$$= 5{\cdot}2 \text{ milliamperes.}$$

I_R is in phase with E, and I_L lags by 90° on E. Therefore I lags on E by an angle $\theta < 90°$;

$$tan\ \theta = \frac{I_L}{I_R} = \frac{3{\cdot}98}{3{\cdot}33} = 1{\cdot}195$$

whence $\theta = 50°$ approximately.

Or :—

$$G = \frac{1}{R} = {\cdot}001667$$
$$B_L = \frac{1}{\omega L} = {\cdot}00199$$
$$Y = \sqrt{G^2 + B_L^2}$$
$$= \frac{\sqrt{2{\cdot}77 + 3{\cdot}95}}{1{,}000} = \frac{2{\cdot}6}{1{,}000}$$
$$I = YE = \frac{2{\cdot}6}{1{,}000} \times 2 = {\cdot}0052 \text{ amperes or } 5{\cdot}2 \text{ milliamperes.}$$
$$\text{and } \tan\theta = \frac{B_L}{G} = \frac{600}{502} = 1{\cdot}195$$

Impedance and resistance in parallel

46. The circuit of fig. 36 shows a resistance of 2 ohms, having an inductance of 0·4 microhenries, in parallel with a purely resistive path of 2 ohms. At very low frequencies, the

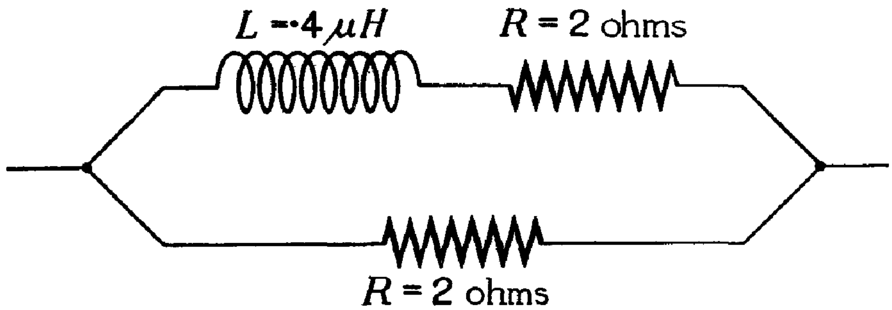


FIG. 36, CHAP. V.—Inductive resistance and non-inductive resistance in parallel.

inductive reactance is negligible and the current in each branch will be the same, for a given terminal P.D. Suppose however that the frequency is fairly high, say 796 kc/s. The inductive reactance is then also 2 ohms, and the impedance of this branch becomes $\sqrt{8}$ ohms, so that the current will be only ·707 of that in the purely resistive branch. The disparity between the two currents will increase with the frequency, and it may be said that where parallel paths are available, a high frequency current will divide inversely as the inductance of the respective paths, the resistance having negligible effect upon the current distribution.

This leads to an alternative view of the cause of skin effect. The centre portion of the cross-section of a conductor is surrounded by a greater number of tubes of flux that is the outer portion, and the peak value of the rate of change of flux increases from the surface to the centre, so that the inductance of the centre is greater than that of the outer portion. As the current divides in inverse ratio to the inductance, it follows that less current will flow in the centre of the cross-section than on the surface.

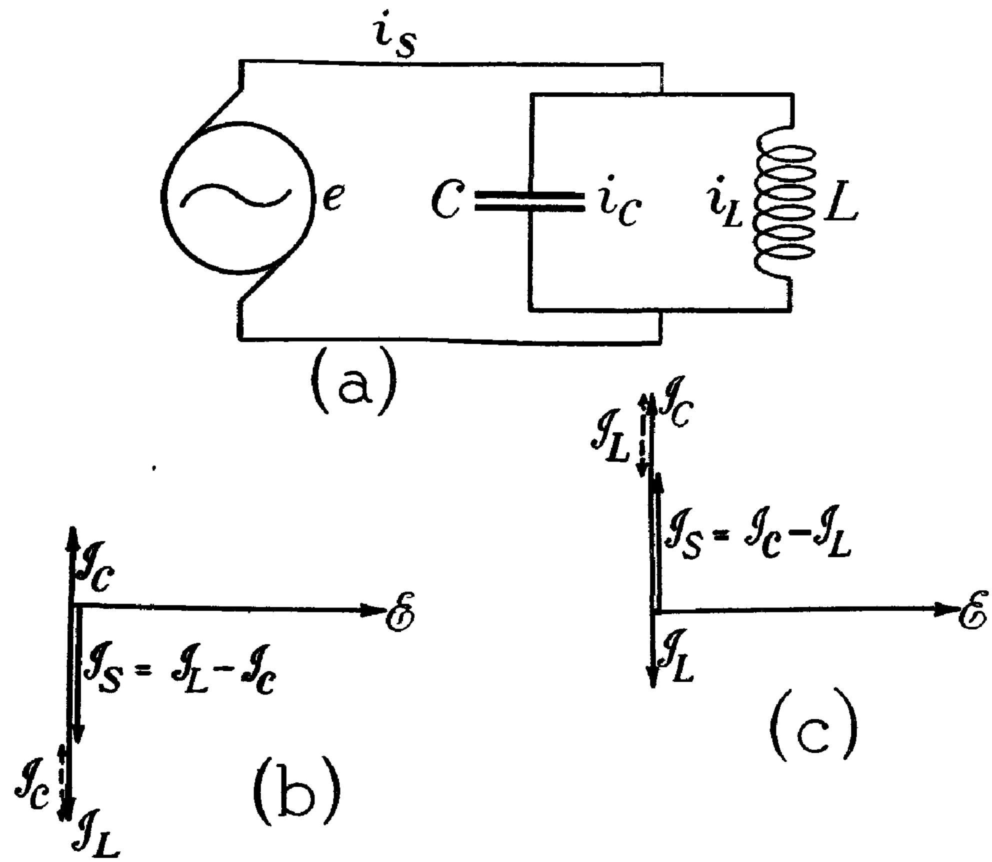


FIG. 37, CHAP. V.—A.C. circuit possessing inductance and capacitance in parallel.

Inductance and capacitance in parallel

47. It is next proposed to consider a circuit in which an inductance L and a condenser C are placed in parallel, and an alternating E.M.F. applied to the parallel combination. In the preliminary investigation, the effects of resistance will be neglected. It must not be supposed however that the results obtained in this way are valueless, for in practical circuits the resistance can often be reduced to a very small value. The circuit under consideration is shown in fig. 37a, in which the alternator E.M.F. has a peak value $\mathcal{E}$, while its frequency is variable. The peak value of the current flowing through the inductance will be $\mathcal{I}_L = \frac{\mathcal{E}}{\omega L}$ lagging on the applied

voltage by 90°, while the peak value of the current charging the condenser will be $\mathcal{I}_C = \omega C\mathcal{E}$, leading on the applied voltage by 90°. The peak value of the supply current $\mathcal{I}_S$ from the alternator will be the vector sum of the currents through the inductance and the condenser. In figs. 37, (*b*) and (*c*) are shown the different conditions which may obtain according to the frequency of the supply. If $\mathcal{I}_L$ is greater than $\mathcal{I}_C$ the supply current $\mathcal{I}_S$ will lag on the applied voltage by 90°, while if $\mathcal{I}_C$ is greater than $\mathcal{I}_L$ the supply current will lead on the applied voltage by 90°. It will be observed that as $\mathcal{I}_C$ and $\mathcal{I}_L$ are 180° out of phase with each other, the vector sum of these is actually their numerical difference; this may be written $\mathcal{I}_S = | \mathcal{I}_C - \mathcal{I}_L |$ the vertical lines indicating that the numerical value of the expression is denoted. As

$$\mathcal{I}_L = \frac{\mathcal{E}}{\omega L}\ \mathcal{I}_C = \omega C\mathcal{E},\ \mathcal{I}_S = | \mathcal{I}_C - \mathcal{I}_L |,$$

$$\mathcal{I}_S = \mathcal{E}\left(\omega C - \frac{1}{\omega L}\right)$$

If the value of $\omega C - \dfrac{1}{\omega L}$ is positive, the current leads on the voltage; if negative the current lags.

Example 10.—An inductance of 160 microhenries and a capacitance of ·0002 microfarad are connected in parallel, the losses in both components being negligible. If an E.M.F. of 100 volts R.M.S. at 800 kc/s is applied, find the R.M.S. supply current and whether lagging or leading.

$$I_S = E\left(\omega C - \frac{1}{\omega L}\right)$$

$$\omega C = 6{\cdot}28 \times 800 \times 1{,}000 \times {\cdot}0002 \times 10^{-6}$$

$$= {\cdot}001005$$

$$\frac{1}{\omega L} = \frac{10^6}{6{\cdot}28 \times 800 \times 1{,}000 \times 160}$$

$$= \frac{1}{804}$$

$$= {\cdot}001245$$

$$\omega C - \frac{1}{\omega L} = {\cdot}001005 - {\cdot}001245$$

$$= -{\cdot}00024$$

$$I_S = 100 \times (-{\cdot}00024)$$

$$= -{\cdot}024 \text{ amperes}$$

Since I_S is negative the current lags on the voltage by 90°. The effective reactance of the parallel combination is therefore inductive, and the value of the equivalent inductance is easily found.

$$X_e = \frac{E}{I_S}$$

$$L_e = \frac{E}{\omega I_S}$$

$$\frac{E}{I_S} = \frac{100}{{\cdot}024}$$

$$= 4{,}160$$

$$L_e = \frac{4{,}160}{6{\cdot}28 \times 800 \times 1{,}000} \times 10^6\ (\mu H)$$

$$= 830\ \mu H$$

Parallel resonance

48. In the particular case when $\omega C = \frac{1}{\omega L}$ the current $\mathcal{J}_C$ is equal to the current $\mathcal{J}_L$ and their arithmetical difference is zero, i.e. $\mathcal{J}_S = 0$. The circuit will therefore take no current whatever from the alternator, and the total effective impedance of the circuit is infinitely great. This condition is not possible in practice because both inductive and capacitive branches must possess some resistance, but may be closely approached. Assuming the resistance to be absolutely negligible, the frequency at which the supply current is zero is easily derived.

$$\text{When} \qquad \omega C = \frac{1}{\omega L}$$

$$\omega^2 = \frac{1}{LC}$$

$$\text{and } f = \frac{1}{2\pi\sqrt{LC}}$$

This is the resonant frequency of the circuit, which is said to be a rejector for this particular frequency. The conditions of parallel resonance are illustrated in fig. 38. Referring to the circuit diagram, at any instant when the current in the inductance is flowing downwards, the

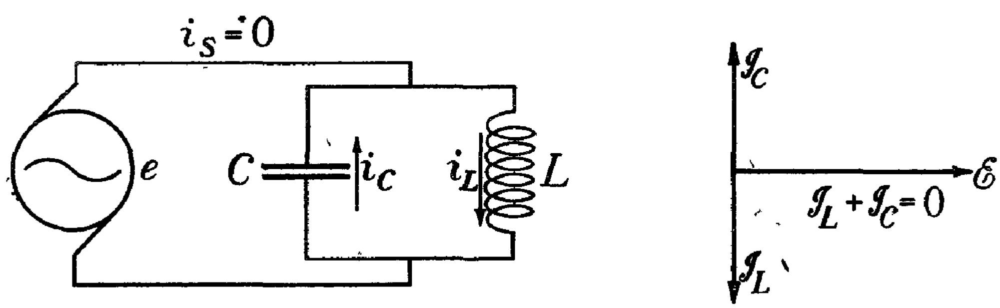


FIG. 38, CHAP. V.—Currents in circuit of Fig. 37, at resonant frequency.

current in the capacitance, which is 180° out of phase with it, must be flowing upwards, and the effect is that of an alternating current flowing to and fro in the closed circuit comprised by the inductance and condenser. For this reason, the current $I_L = I_C$ is often called the (R.M.S.) circulating current; its value depends upon the supply voltage and upon the ratio of capacitance to inductance.

$$\text{Since} \qquad I_L = \frac{E}{\omega L} \text{ and } I_C = \omega CE$$

$$I_L \times I_C = I_L^2 \text{ or } I_C^2$$

$$I_L \times I_C = \frac{E}{\omega L} \times \omega CE = \frac{C}{L} E^2$$

$$\therefore\ I_L \text{ and } I_C = E\sqrt{\frac{C}{L}}$$

The current I_L is greater than I_C when $\frac{1}{\omega L}$ is greater than ωC, that is at frequencies below the resonant frequency. The supply current then lags by 90° on the applied voltage. On the other hand, at frequencies above resonance, ωC is greater than $\frac{1}{\omega L}$ and I_C is greater than I_L. The supply current will then lead on the applied voltage by 90°.

The effect of a small resistance in a rejector circuit

49. In practice, every circuit must contain some resistance however small, and it is now intended to consider the effect upon the action of a rejector circuit of a resistance R_L of the order which would be encountered in a practical, efficient circuit. Suppose the resistance to exist only in the inductive branch of the circuit. Remembering that $\chi_L = \frac{\omega L}{R_L}, = \tan \theta_L$ where θ_L is the angle of lag of $\mathcal{I}_L$, reference to a table of tangents shows that if $\chi_L = 50$, $\mathcal{I}_L$ will lag on the applied voltage by very nearly 89°, while if $\chi_L = 100$, $\mathcal{I}_L$ will lag by nearly 89·5°. As the condenser is supposed to have no losses, the current $\mathcal{I}_C$ will lead on the applied voltage by 90°.

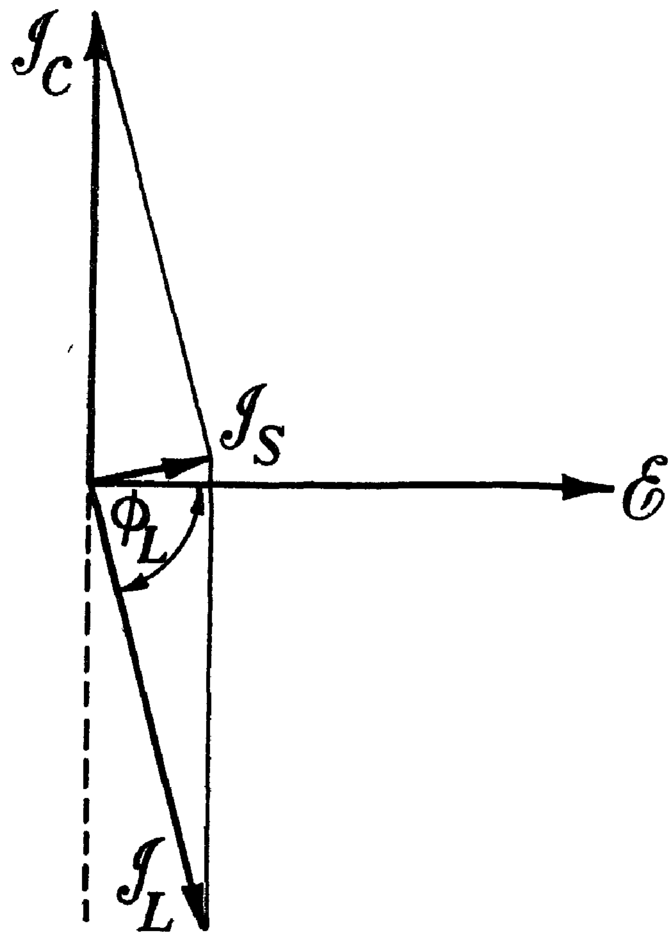


FIG. 39, CHAP. V.—Make-up current in rejector circuit possessing resistance.

$\mathcal{I}_L$ and $\mathcal{I}_C$ are thus practically 180° out of phase with each other, and at any frequency except the resonant frequency the R.M.S. value of the supply current (I_s) is to all intents and purposes the difference between $I_L \left(= \frac{\mathcal{I}_L}{\sqrt{2}}\right)$ and $I_C \left(= \frac{\mathcal{I}_C}{\sqrt{2}}\right)$.

At the resonant frequency the power expended in the circuit will be $I_s^2 R_L$ or $\frac{\mathcal{I}_s^2 R_L}{2}$ watts, but $\mathcal{I}_s$ is not the arithmetic difference between $\mathcal{I}_L$ and $\mathcal{I}_C$ because $\mathcal{I}_s$ is now practically in phase with the applied voltage. The vector diagram fig. 39 has been drawn to explain this, although the angle φ_L has been shown as much less than 89° for clearness. The value of the supply current must depend upon the power expended in the circuit. Now this power is $I_L^2 R_L$ (or $I_C^2 R_L$ because $I_L^2 = I_C^2$) therefore the power expended is $I_L I_C R_L$ watts or

$$P = \frac{E}{\omega L} \times \omega CE \times R_L$$

$$= \frac{CR_L}{L} E^2 \text{ watts.}$$

As the supply current is in phase with the supply voltage, the circuit as a whole must be acting as a resistance. There is nothing new in this, for we saw that the acceptor circuit at its resonant frequency behaved as if it contained resistance only. Now in any circuit whatever the power expended may be expressed as $\frac{E^2}{R_d}$ where E is the R.M.S. value of the applied E.M.F. and R_d is the effective resistance of the circuit. The power $\frac{CR_L}{L} E^2$ may therefore be equated to $\frac{E^2}{R_d}$, giving the effective resistance of the circuit;

$$\frac{E^2}{R_d} = \frac{E^2 CR_L}{L}$$

$$\therefore R_d = \frac{L}{CR_L}$$

The effective resistance of the rejector circuit at its resonant frequency is frequently referred to as its dynamic resistance. The R.M.S. value, I_s, of the supply current is $\frac{E}{R_d}$ or $\frac{ECR}{L}$ amperes, if E is in volts. It must be particularly borne in mind that the greater the actual resistance of the circuit the less is its dynamic resistance.

Resonance curves for a rejector circuit

50. Let us now take a rejector circuit containing a small resistance, and vary the frequency of the applied E.M.F. as was done in the study of an acceptor circuit, keeping its R.M.S. value constant, say 10 volts. Suppose that $L = 1 \cdot 6\ \mu H$, $C = \cdot 025\ \mu F$, and $R = \cdot 064$ ohms. Then as the frequency is varied from 764 to 828 kc/s the corresponding variation of current is shown by curve (i) of fig. 40. Such a curve is called the resonance curve of the rejector circuit. It will be observed that at the resonant frequency, 796 kc/s, the current falls to a value $\frac{ECR}{L} = \cdot 01$ ampere, rising on each side of the resonant frequency. The rate at which the current increases, as the frequency is varied above or below the resonant frequency, depends upon the ratio $\frac{L}{R}$, as in the acceptor circuit. Thus if R remains constant, the effect of an alteration in the ratio $\frac{L}{C}$ is shown by curves (ii) and (iii). In curve (ii) R is $\cdot 064$ ohms, but $L = 3 \cdot 2\ \mu H$ and $C = \cdot 0125\ \mu F$, while in curve (iii) R is $\cdot 064$ ohms, $L = 1 \cdot 13\ \mu H$, $C = \cdot 0353\ \mu F$. Now it will be observed that the current at any non-resonant frequency is greater in curve (iii) than in curve (i) or (ii) and it is therefore often erroneously concluded that the " selectivity " of a rejector circuit is increased as the ratio $\frac{C}{L}$ is increased. Before proceeding further, it must be pointed out that when speaking of a circuit consisting of an inductance and condenser in parallel, the term selectivity must be used with some caution. If the circuit is used as a true rejector, that is to suppress current at one particular frequency, then the criterion of its " selectivity " is the ratio (current passed by the device at the desired frequency) over (current passed at an undesired frequency). In fig. 40 the undesired frequency would be 796 kc/s, while we may suppose that the desired frequency i.e. that which is required to pass through the rejector circuit, is 780 kc/s. Then in curve (i) the

ratio of desired (I_n) to undesired, (I_r), currents is 5·3 to 1, whereas in curve (ii) it is = 10 to 1, and in curve (iii) it is 3·6 to 1. The circuit corresponding to curve (ii) is thus the best of the three circuits as a " wave trap " as this form of current suppressor is generally called.

Now let us study the effect of varying R, while keeping L and C constant. Suppose that we take the original circuit, $L = 1\cdot 6\ \mu H$, $C = \cdot 025\ \mu F$ but reduce R to ·032 ohms. Then the resulting resonance curve is shown in fig. 41 curve (ii), curve (i) being repeated from the previous figure to serve as a basis of comparison. It is again obvious that taking the ratio I_n/I_r, or (current at any non-resonant frequency) / (current at resonant frequency) as a criterion the selectivity of each circuit is proportional to the ratio $\frac{L}{R}$. The greater this ratio is, the greater the proportion of current at non-resonant frequency to the current at resonance.

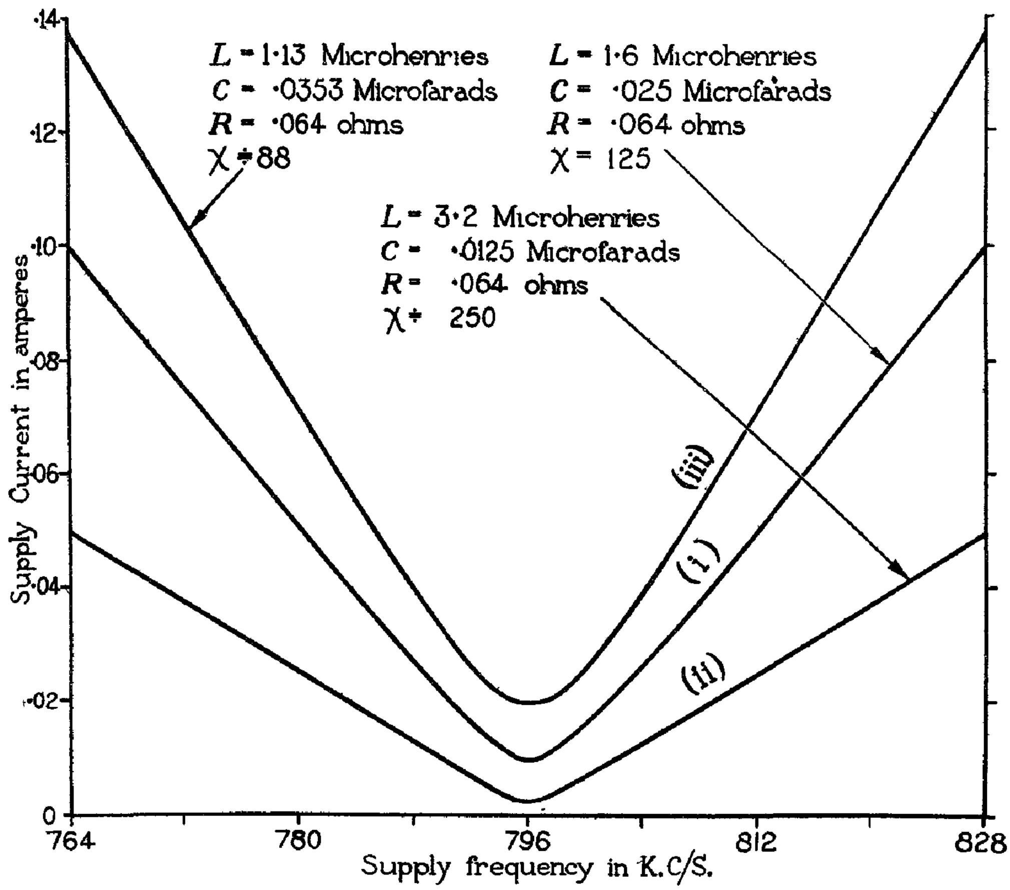


FIG. 40, CHAP. V.—Parallel resonance curves. Effect of ratio of inductance to capacitance.

When a rejector circuit is used in such a manner that the ratio of resonant to non-resonant P.D. at its terminals is of major importance, the above conclusions must be modified. The effect of any impedance which is effectively in parallel with it must be taken into consideration and it is desirable to analyse any particular case from first principles rather than to rely on general conclusions. An example of such an analysis is given in paragraph 64.

Circuit magnification of a rejector circuit

51. It should be obvious that a circuit consisting of L and C in parallel, whether inherent resistance is present or not, cannot have a voltage magnification, for the voltage across the inductance or condenser cannot be greater than the applied voltage. Instead the magnification

takes the form of current magnification that is, the circulating current is χ times as great as the supply current. This is easily shown as follows. The R.M.S. circulating current, is equal to $\frac{E}{\omega L}$ or $\omega C\, E$, while the supply current I_s is equal to $\frac{CR}{L}\, E$. The ratio I_L/I_s is therefore

$$\frac{E}{\omega L} \div \frac{CR}{L} E$$

$$= \frac{E}{\omega L} \times \frac{L}{CRE} = \frac{1}{\omega CR} = \chi$$

$$\text{or} \qquad \omega CE \times \frac{L}{CRE} = \frac{\omega L}{R} = \chi$$

and the statement that $I_L = \chi\, I_s$ is proved. This relation can often be used to shorten work in connection with actual radio circuits.

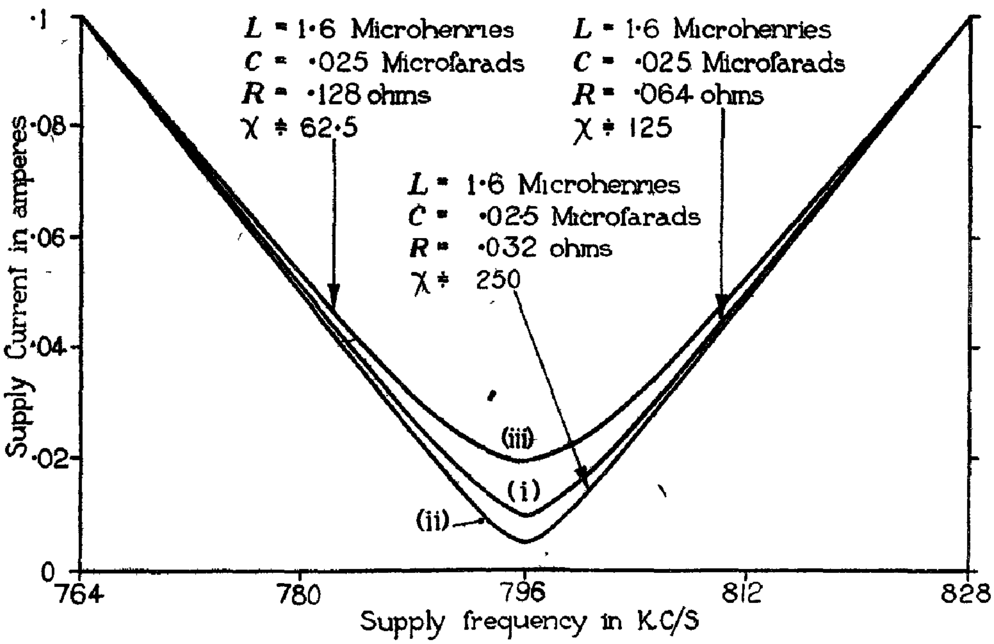


FIG. 41, CHAP. V.—Parallel resonance curves. Effect of variation of resistance.

Rejector circuit with considerable resistance

52. Although the circumstances rarely occur in radio circuits, it is desirable from a purely theoretical point of view to consider the conditions arising in a parallel inductance-capacitance combination when the resistance of either or both branches is comparable in magnitude with the reactance of the branch. Let us assume therefore that a circuit consists of an inductance of L henries which has a resistance of R_L ohms and a condenser of capacitance C farads, the losses in which can be represented by a resistance of R_0 ohms. These being connected in parallel, an alternating E.M.F. of peak value $\mathscr{E}$ volts, and of variable frequency, is applied. The current in the inductance will be $\mathscr{I}_L = \frac{\mathscr{E}}{Z_L}$ where $Z_L = \sqrt{R^2 + (\omega L)^2}$ and will lag by an angle $\varphi_L = tan^{-1}\frac{\omega L}{R_L}$ on the applied voltage. The current charging the condenser will be $\mathscr{I}_0 = \frac{\mathscr{E}}{Z_0}$ where $Z_0 = \sqrt{R_0^2 + \left(\frac{1}{\omega C}\right)^2}$ and will lead by an angle $\varphi_0 = tan^{-1}\frac{1}{\omega C R_0}$. The vector diagram of the

resulting conditions is given in fig. 42. The supply current $\mathcal{I}_S$ is given by the vector sum of $\mathcal{I}_L$ and $\mathcal{I}_C$, and is obtained by completing the parallelogram two sides of which are $\mathcal{I}_L$ and $\mathcal{I}_C$. The diagonal through the origin is then equal to the supply current.

In order to find an algebraic expression for the supply current, in terms of the applied E.M.F. and the circuit constants, it is necessary to resolve $\mathcal{I}_L$ and $\mathcal{I}_C$ each into two components in phase with, and 90° out of phase with the voltage. The components of $\mathcal{I}_L$ are (i) $\mathcal{I}_L \sin \varphi_L$, 90° out of phase, and (ii) $\mathcal{I}_L \cos \varphi_L$ in phase, with the applied E.M.F. φ_L. If any doubt is felt as to the justification for this procedure, it may be dispelled by confirming that the vector sum of $\mathcal{I}_L \sin \varphi_L$ and $\mathcal{I}_L \cos \varphi_L$ is $\mathcal{I}_L$. Now the vector sum of these is $\sqrt{(\mathcal{I}_L \cos \varphi_L)^2 + (\mathcal{I}_L \sin \varphi_L)^2}$ and since $(\cos a)^2 + (\sin a)^2$ where a is an angle whatever, is equal to unity, we may consider the proposition proved. In the same way, the components of $\mathcal{I}_C$ are (i) $\mathcal{I}_C \sin \varphi_C$, 90° out of phase, and (ii) $\mathcal{I}_C \cos \varphi_C$ in phase, with the voltage.

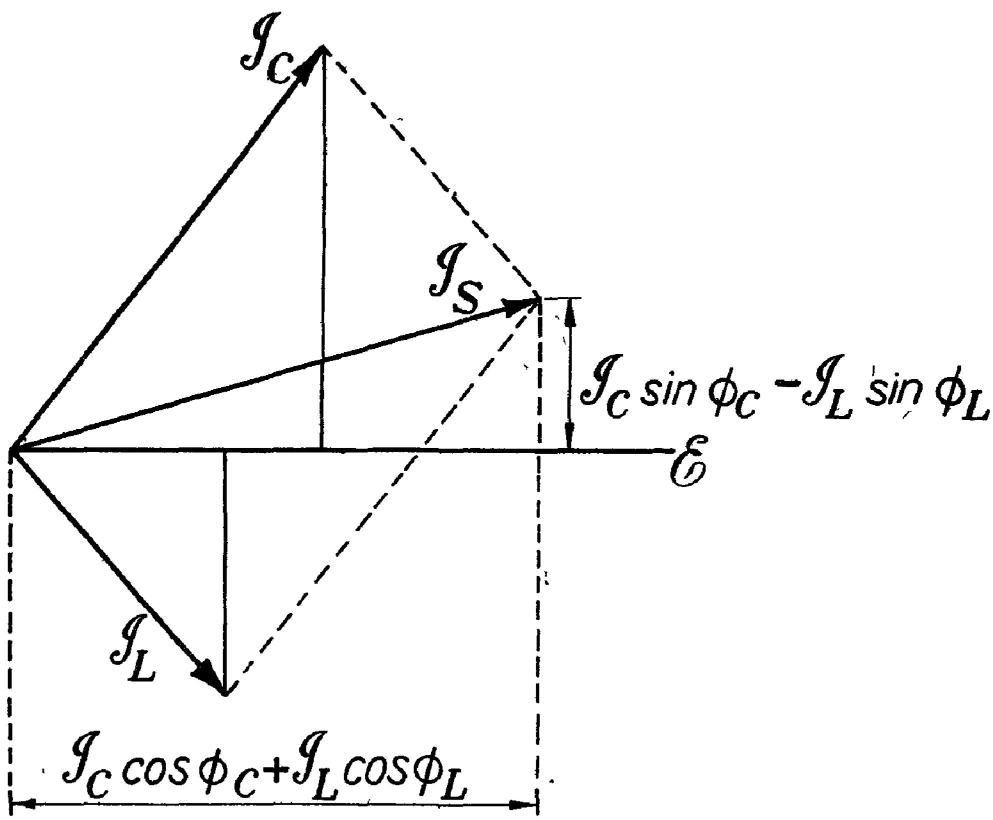


FIG. 42, CHAP. V.—Vector diagram showing relation between currents in rejector circuit possessing considerable resistance.

Now consider the vector $\mathcal{I}_S$ representing the supply current. $\mathcal{I}_S$ can also be divided into in-phase and 90° out-of-phase components; its out-of-phase component is equal to the difference between $\mathcal{I}_L \sin \varphi_L$ and $\mathcal{I}_C \sin \varphi_C$, while its in-phase component is equal to the sum of $\mathcal{I}_L \cos \varphi_L$ and $\mathcal{I}_C \cos \varphi_C$. It should be noted that this is true even if the circuit possesses no resistance, for in this instance $\sin \varphi_L$ and $\sin \varphi_C$ are each equal to unity because both inductive and capacitive currents are 90° out of phase and $\sin 90° = 1$, while $\cos \varphi_L$ and $\cos \varphi_C$ are each equal to zero because $\cos 90° = O$. Hence it may be said that in all cases

$$\mathcal{I}_S = \sqrt{\{(\mathcal{I}_L \sin \varphi_L - \mathcal{I}_C \sin \varphi_C)^2 + (\mathcal{I}_L \cos \varphi_L + \mathcal{I}_C \cos \varphi_C)^2\}}$$

In order to eliminate the trigonometrical terms from this equation we use the relation

$$\sin \varphi_L = \frac{X_L}{Z_L} \qquad \sin \varphi_C = \frac{X_C}{Z_C}$$

$$\cos \varphi_L = \frac{R_L}{Z_L} \qquad \cos \varphi_C = \frac{R_C}{Z_C}$$

giving

$$\mathcal{I}_s = \sqrt{\left(\frac{\mathcal{E}}{Z_L} \times \frac{X_L}{Z_L} - \frac{\mathcal{E}}{Z_C} \times \frac{X_C}{Z_C}\right)^2 + \left(\frac{\mathcal{E}}{Z_L} \times \frac{R_L}{Z_L} + \frac{\mathcal{E}}{Z_C} \times \frac{R_C}{Z_C}\right)^2}$$

$$= \mathcal{E} \sqrt{\left(\frac{X_L}{Z_L^2} - \frac{X_C}{Z_C^2}\right)^2 + \left(\frac{R_L}{Z_L^2} + \frac{R_C}{Z_C^2}\right)^2}$$

The expression under the square root sign is the admittance of the parallel circuit. The susceptance of the combination is

$$\frac{X_L}{Z_L^2} - \frac{X_C}{Z_C^2} = B,$$

and the conductance is

$$\frac{R_L}{Z_L^2} + \frac{R_C}{Z_C^2} = G$$

The supply current will be in phase with the supply voltage when the susceptance of the circuit is zero. This occurs when

$$\frac{X_L}{Z_L^2} = \frac{X_C}{Z_C^2}$$

and the frequency at which this equation is true is called the resonant frequency of the circuit.

This frequency is found by expressing the equation is such a way that the frequency, or ω which is 2π times the frequency appears thus :—

$$\frac{\omega L}{R_L^2 + \omega^2 L^2} = \frac{\dfrac{1}{\omega C}}{R_C^2 + \dfrac{1}{\omega^2 C^2}}$$

$$\frac{\omega L}{R_L^2 + \omega^2 L^2} = \frac{\omega C}{\omega^2 C^2 R_C^2 + 1}$$

$$\frac{L}{R_L^2 + \omega^2 L^2} = \frac{C}{\omega^2 C^2 R_C^2 + 1}$$

cross multiplying,

$$\omega^2 L C^2 R_C + L = R_L^2 C + \omega^2 L^2 C$$

Collecting terms containing ω to the left hand side.

$$\omega^2 (LC^2 R_C^2 - L^2 C) = R_L^2 C - L$$

whence

$$\omega^2 = \frac{R_L^2 C - L}{LC^2 R_C^2 - L^2 C}$$

$$= \frac{1}{LC} \left\{\frac{R_L^2 C - L}{C R_C^2 - L}\right\}$$

$$\omega = \sqrt{\frac{1}{LC} \left\{\frac{R_L^2 C - L}{C R_C^2 - L}\right\}}$$

Finally

$$f = \frac{1}{2\pi} \sqrt{\frac{1}{LC} \left\{\frac{R_L^2 C - L}{C R_C^2 - L}\right\}}.$$

From these equations it can be seen that only in the particular case when $R_C = R_L$ is the resonant frequency equal to

$$\frac{1}{2\pi\sqrt{LC}}$$

In most circumstances however, the factor enclosed in brackets only differs from unity by a very small fraction.

An interesting effect occurs if the numerator and denominator of the bracketed portion of the equation both become equal to zero. The " resonant frequency " according to the formula then becomes

$$f = \frac{1}{2\pi}\sqrt{\frac{1}{LC} \times \frac{0}{0}}$$

Now zero is not a number but the absence of any number whatsoever, and when an expression takes the form $\frac{0}{0}$ it is said to be indeterminate. The condition in which the resonant frequency becomes indeterminate is when $R_L = R_C = \sqrt{\frac{L}{C}}$. The circuit then behaves as if each branch has a resistance $\sqrt{\frac{L}{C}}$ in series with its inherent resistance, and the two branches in parallel then have a total joint resistance of $\sqrt{\frac{L}{C}}$ ohms, no matter what the applied frequency may be.

This phenomenon has an analogy in the case of a transmission line which contains inductance and capacitance distributed all along its length, its inductance per unit length being l henries and its capacitance per unit length being c farads. The line will then transmit all frequencies equally well if the "load" at the end of the transmission line consists of resistance only, its value being equal to $\sqrt{\frac{l}{c}}$ ohms.

53. Reverting to the consideration of the admittance Y, of a rejector circuit, we have seen that

$$Y = \sqrt{G^2 + B^2}$$

The latter expression can be considerably simplified for practical use with negligible sacrifice of accuracy, by introducing the figures of merit, χ_L, χ_C thus

$$\frac{X_L}{Z_L^2} = \frac{\omega L}{R_L^2 + \omega^2 L^2} = \frac{\omega L}{\omega^2 L^2\left(1 + \frac{R_L^2}{\omega^2 L^2}\right)} = \frac{\chi_L^2}{\omega L\,(1 + \chi_L^2)}$$

$$\frac{X_C}{Z_C^2} = \frac{\frac{1}{\omega C}}{R_C^2 + \frac{1}{\omega^2 C^2}} = \frac{\omega C}{1 + \omega^2 C^2 R_C^2} = \frac{\omega C\,\chi_C^2}{1 + \chi_C^2}$$

Hence $$B = \omega C\left(\frac{\chi_C^2}{1 + \chi_C^2}\right) - \frac{1}{\omega L}\left(\frac{\chi_L^2}{1 + \chi_L^2}\right)$$

and when χ_L and $\chi_C >> 1$, which is almost always the case in radio circuits, this simplifies to

$$B \doteqdot \omega C - \frac{1}{\omega L}.$$

Similarly
$$\frac{R_L}{Z_L^2} = \frac{1}{R_L\left(1 + \frac{\omega^2L^2}{R_L^2}\right)} = \frac{1}{R_L\,(1 + \chi_L^2)}$$
$$\frac{R_C}{Z_C^2} = \frac{1}{R_C\left(1 + \frac{1}{\omega^2C^2R_C^2}\right)} = \frac{1}{R_C\,(1 + \chi_C^2)}$$

Hence
$$G = \frac{1}{R_L\,(1 + \chi_L^2)} + \frac{1}{R_C\,(1 + \chi_C^2)} \quad .$$

Again if χ_L and $\chi_C >> 1$ this simplifies to
$$G = \frac{1}{R_L\chi_L^2} + \frac{1}{R_C\chi_C^2} \quad ,$$
and if R_C is negligible,
$$G = \frac{1}{R_L\chi_L^2} = \frac{R_L}{\omega^2L^2} \quad .$$

For nearly all practical purposes, then,
$$Y = \sqrt{\left(\omega C - \frac{1}{\omega L}\right)^2 + \left(\frac{1}{R_L\,\chi_L^2} + \frac{1}{R_C\chi_C^2}\right)^2}$$

At frequencies very near to the resonant frequency, the conductance may be considered equal to the conductance at resonance, that is $\frac{1}{R_d}$ or $\frac{CR}{L}$. The admittance is then
$$Y = \sqrt{\left(\omega C - \frac{1}{\omega L}\right)^2 + \left(\frac{CR}{L}\right)^2}$$
the R.M.S. supply current being found by the relation $I = YE$ as usual.

Example 11.—A coil having an inductance of 160 microhenries and a resistance of 19 ohms is connected in parallel with a condenser having a capacitance of ·000256 microfarad and a resistance of 1 ohm. Obtain an approximation to the admittance at the frequency corresponding to $\omega = 5 \times 10^6$, the current set up by an R.M.S. voltage of 100 volts, and the angle of lag or lead.

$$\omega C = \cdot 000256 \times 10^{-6} \times 5 \times 10^6 = \cdot 00128$$
$$\frac{1}{\omega L} = \frac{1}{5 \times 10^6 \times 160 \times 10^{-6}} = \frac{1}{800} = \cdot 00125$$
$$B = \omega C - \frac{1}{\omega L} = \cdot 00003$$

Total resistance = 20 ohms.

$$G = \frac{CR}{L} = \frac{\cdot 000256 \times 20}{160} = \cdot 000032$$
$$\begin{aligned} Y &= \sqrt{G^2 + B^2} \\ &= \sqrt{(32 \times 10^{-6})^2 + (30 \times 10^{-6})^2} \\ &= 10^{-6}\sqrt{1{,}024 + 900} \\ &= 43{\cdot}8 \times 10^{-6} \text{ siemens (or mho)} \\ I = YE &= 43{\cdot}8 \times 10^{-6} \times 100 \\ &= 4{,}380 \text{ microamperes.} \end{aligned}$$

As ωC is greater than $\frac{1}{\omega L}$ the current leads on the applied voltage by an angle $\theta = tan^{-1} \frac{B}{G}$

$$\frac{B}{G} = \frac{\cdot 00003}{\cdot 000032} = \cdot 936$$

$$\therefore \theta \doteqdot 43°$$

VECTOR OPERATORS

54. A stage has now been reached at which the relations between current voltage and impedance in various types of A.C. circuit can be found by the use of vector diagrams in con junction with elementary trigonometry. It is possible to solve practically any A.C. problem without further mathematical knowledge, but many problems are much more easily handled by expressing the vector quantities themselves by symbols denoting not only their magnitude but also their direction. One method which is sometimes employed in connection with impedances is to denote the magnitude of the impedance by a symbol or by a number representing the magnitude followed by a symbol or number representing the angle of phase difference caused by this impedance, for example, (Z, θ) would be an impedance of magnitude Z, causing a phase difference of θ radians, but as this alone would not indicate whether the impedance caused a leading or lagging current, the sign $\angle$ is used to denote an angle of lag, and $\overline{\vee}$ to denote an angle of lead. Thus $Z \underline{/\theta}$ is equivalent to $\sqrt{R^2 + \omega^2L^2}$, where $\frac{\omega L}{R} = tan^{-1} \theta$. $600 \underline{/\frac{\pi}{2}} = \sqrt{R^2 + \omega^2L^2}$, where $\frac{\omega L}{R} = tan^{-1} \frac{\pi}{2}$; but $tan^{-1} \frac{\pi}{2} = \infty$, and $\frac{\omega L}{R}$ is consequently infinite also. As ωL cannot be infinite, because $\sqrt{R^2 + \omega^2L^2}$ is only 600 ohms, R must be zero, and therefore $600 \underline{/\frac{\pi}{2}}$ represents and inductive reactance of 600 ohms. A further example may be given: let $Z \underline{/\theta} = 800 \underline{/\cdot 5326}$, or $800 \underline{/30°}$ $\cdot 05326$ being the radian measure of 30°.

$$\text{Then } \sqrt{R^2 + \omega^2L^2} = 800$$

$$\text{and} \quad \frac{\omega L}{R} = tan\ 30° = \cdot 5774 \text{ or } \frac{1}{\sqrt{3}}$$

$$\frac{\omega^2L^2}{R^2} = \frac{1}{3}$$

$$\omega^2L^2 = \frac{R^2}{3}$$

$$\text{Also} \quad R^2 + \omega^2L^2 = 6{,}400$$

$$R^2\left(1 + \frac{1}{3}\right) = 6{,}400$$

$$R^2 = \frac{6{,}400 \times 3}{4} = 4{,}800$$

$$\therefore R = 69 \cdot 2 \text{ ohms,}$$

$$\text{while } \omega^2L^2 = \frac{4{,}800}{3} = 1{,}600$$

$$\text{and} \quad \omega L = 40 \text{ ohms.}$$

This method of indicating the nature of an impedance of a given magnitude is frequently employed in telephone engineering.

55. A method which leads itself more readily to manipulation is that which introduces the conception of a " vector operator." In order to explain this, it may be desirable to consider the idea of a mathematical operation. When a quantity is denoted by $a + b$, the plus sign may be considered to denote an operation performed upon b, for we may consider that originally there were two groups containing a objects and b objects respectively, and an operation is performed upon the latter group, the whole being conveyed to and possibly intermingled with the former group so that a new group is formed, containing a number of objects equal to the sum of the numbers in the two original groups. If instead of $a + b$, we write $b + a$, we may consider that the group denoted by b is stationary and that the operation of addition is performed by moving the group a. The numerical result is of course the same in each case, and $a + b = b + a$.

The plus sign thus indicates the operation of addition, while the minus sign when used in arithmetic signifies the operation of subtraction, e.g. $a - b$ means that from a group of a objects a number b is taken. The minus sign is also called the negative sign, for it is also used to convey the conception of contrariety of direction with regard to a given direction which is arbitrarily said to be positive. The two meanings of the signs + and — rarely if ever give rise to confusion. If a certain problem concerns objects, the equation $7 - 3 = 4$, for example, indicates that the operation of subtraction is performed upon three members of a group of seven, and the residue will consist of four members. The expression $3 - 7 = -4$ is meaningless when applied to objects, but is intelligible when applied to distances in space if movement in a certain direction is assumed to be positive and movement in a direction exactly opposite to this to be negative, for the equation may then be considered to signify that, starting from a point which may be denoted by zero (0) an object is moved through three units of distance in the positive direction, and afterwards through seven units in the contrary or negative direction, its final position being four units from the original position (or origin) in the negative direction.

56. Two symbols used in conjunction with numbers may also require definition. The symbol 0 (zero) signifies the absence of any number, while the symbol ∞ (infinity) is used to denote a whole series of numbers which are greater than it is possible to comprehend. It must not be thought that the sign ∞ denotes a single, extremely large number. The untruth of this can be seen by allowing ∞ to represent some number too great to admit of comprehension, and then considering it to be raised to the power n, where n is a number greater than unity. The result of this operation is a number larger than the original one, which was denoted by ∞, but as it is too large for comprehension we still denote it by ∞, which thus becomes a symbol denoting a range of numbers and not a single number. This leads to another conception of zero, for if infinity is defined as some number larger than can be conceived $\frac{1}{\infty}$ must be some number smaller than admits of conception, and this exceedingly small quantity $\frac{1}{\infty}$ is denoted by zero, hence the sign 0 has a dual existence, sometimes signifying the entire absence of any number and sometimes a quantity smaller than it is possible to imagine. We see, therefore, that many of the signs used in ordinary arithmetic and algebra are capable of different interpretations, yet it is rare that any confusion arises as to their meaning in any given example.

When applied to distances in space, the symbol $+\infty$ represents an infinitely large distance in the positive direction, and $-\infty$ some infinitely large distance in the opposite direction. The whole series of numbers from $+\infty$ through zero to $-\infty$ with the exception of zero itself, are termed real numbers. This qualification " real " arose from the impossibility of representing the square root of a negative quantity by a number either positive or negative, because no quantity when multiplied by itself will give a negative quantity. Quantities like $\sqrt{-4}$, $\sqrt{-b}$, etc., were therefore called imaginary to distinguish them from quantities like $\sqrt{4}$, $\sqrt{6{\cdot}28}$, etc., the square root of which can be determined with an error smaller than any assignable magnitude if sufficient decimal places are calculated. Any imaginary quantity can be expressed as the product of $\sqrt{-1}$ and a real quantity e.g.

$$\sqrt{-4} = \sqrt{(-1) \times 4} = \sqrt{-1} \times \sqrt{4} = \sqrt{-1} \times 2$$

The term $\sqrt{-1}$ is generally denoted in mathematical textbooks by the symbol i, but in electrical literature by the symbol j to avoid confusion with the symbol for current. The choice of the term " imaginary " to denote a quantity such as $\sqrt{-1} \times 2$ or $j2$ is unfortunate because it may lead to the impression that no meaning can be assigned to such a quantity, whereas it will be shown that " j " may be regarded as a symbol of operation.

57. We have seen that real numbers can be represented by distances from the origin along an arbitrary axis. Conventionally, the positive direction of this axis extends for an infinite distance to the right of the origin, while the negative direction of this axis extends for an infinite distance to the left of the origin. The multiplication of a positive quantity by -1 can therefore be considered as its reversal in direction, by rotation through 180° or π radians in either a clockwise or anticlockwise direction. Again, the result of multiplying $+a$ by $-b$ is the quantity $-ab$, which can be regarded as a multiplication of $+a$ by $+b$, giving a quantity of magnitude ab, and a rotation through 180° as in the previous instance. A further multiplication by -1 may be regarded as a second rotation of 180°, so that -1×-1 is equivalent to a single rotation of 360° or 2π radians. An operator which rotates a vector quantity in this way is termed a versor.

The quantity $\sqrt{-1}$ is a number which if multiplied by itself gives -1. The application of $\sqrt{-1}$ twice in succession to the quantity a gives $\sqrt{-1}\sqrt{-1}\,a$ or $-a$, and is equivalent to a rotation of a through 180° or two right-angles. Now let us postulate an operator which will effect a rotation through only one right-angle; this operator may be termed a quadrantal versor. For convenience we may also define a unit vector as one having unit length and lying in the positive direction along the axis of real numbers. Then operating with the quadrantal versor upon the unit vector twice in succession will result in turning the latter through two right angles, which is π radians or 180°, that is two successive operations by the quadrantal versor are equivalent to a single multiplication by -1, or by two successive multiplications by $\sqrt{-1}$, because $\sqrt{-1} \times \sqrt{-1} = -1$. The result of a single operation by the quadrantal versor therefore appears to be identical with that achieved by a multiplication by the factor $\sqrt{-1}$, or j. By convention the positive direction of rotation is anticlockwise. If the operation denoted by j is described as " jaying " the vector, we see that commencing with a unit vector, jaying the latter once results in a rotation through 90°, jaying twice a rotation through 180°, because $j \times j = j^2 = -1$. A further operation gives $j \times j \times j = j\,j^2 = -1 \times j$ or $-j$, while a fourth gives $j \times j \times j \times j = j^2 \times j^2 = -1 \times -1 = +1$, and the unit vector becomes a unit vector once more,

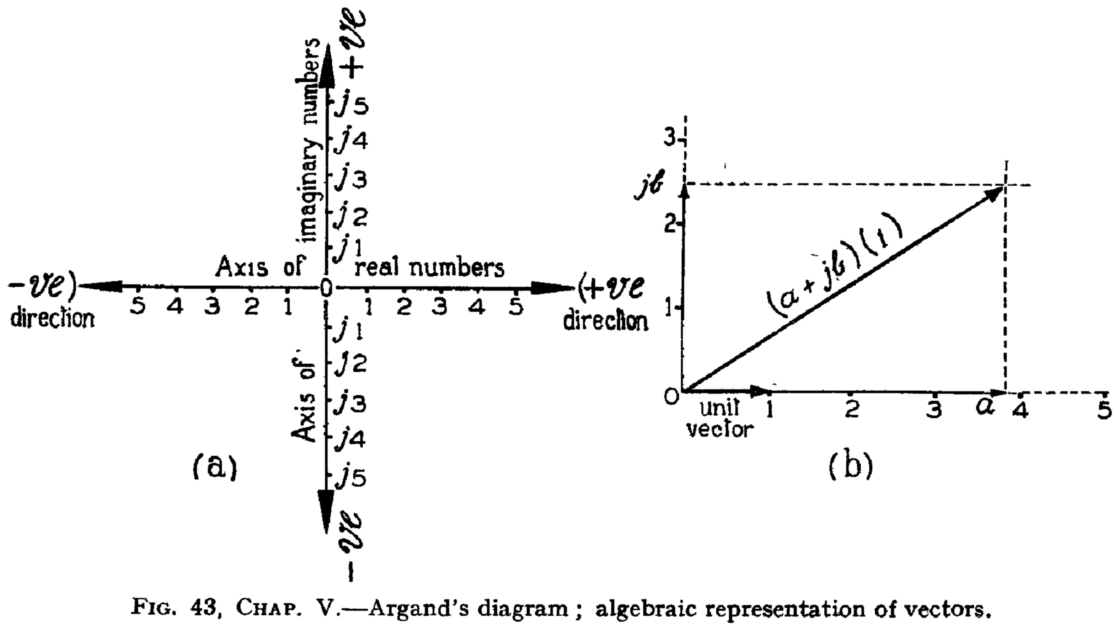


FIG. 43, CHAP. V.—Argand's diagram; algebraic representation of vectors.

having its original length and direction. With reference to the axis of real quantities the axis of a quantity upon which j has operated once only, is vertically upward from the origin, while the axis of those upon which $-j$ has operated once only is vertically downward through the origin. This may be shown upon what is known as Argand's diagram (fig. 43a). The fact that $\sqrt{-1}$ appears to lead a dual existence, sometimes appearing merely as an imaginary number and at others as a quadrantal versor need cause no confusion.

58. Complex numbers are those which consist of the sum of a real and imaginary number, e.g. $a + jb$. If two such quantities are equal to each other, e.g.

$$a + jb = c + jd$$

the real parts a and c must be equal to each other and the imaginary parts jb and jd also equal, because it is impossible for an imaginary number to be equal to a real number by definition.

Thus if $a + jb = c + jd$

$$(a + jb) - (a + jb) = c + jd - (a + jb)$$
$$0 = c - a + j(d - b)$$
$$\therefore c = a \text{ and } d = b$$

unless $a - c = j(d - b)$, i.e. a real number equal to an imaginary one, which is impossible.

A complex number can be an operator. Suppose that the number $(a + jb)$ operates upon a unit vector, which may be denoted by (1). Then $(a + jb)\ (1) = a\ (1) + jb(1)$. The result of the operation is the sum of two vectors one of which is in the direction of unit vector but is a times the magnitude of the latter, while the other is b times the magnitude of the unit vector but has been "jayed" i.e. rotated through 90° in the anticlockwise direction. The magnitude of the vector $(a + jb)\ (1)$ is seen from its diagrammatic representation (fig. 43b) to be $\sqrt{a^2 + b^2}$. The angle through which the vector has rotated with reference to the direction of the unit vector is $tan^{-1} \frac{b}{a}$. When used in conjunction with vector operators it is usual to denote vector quantities by clarendon type. The original vector operated upon may be an alternating current, e.g. **I**, and the operator $R + j\omega L$. Then $(R + j\omega L)\ \mathbf{I}$ is a vector which may be regarded as the sum of two vectors, viz. $R\,\mathbf{I} + j\omega L\,\mathbf{I}$ the first being an E.M.F. which is in phase with the current and the second an E.M.F. which leads on the current by 90°. The magnitude of the sum of these is $\sqrt{R^2 + \omega^2 L^2}\ I$ where I is the magnitude of the current **I**, and the above result may be written symbolically as

$$\mathbf{e} = Z\,\mathbf{I}$$
$$\text{or } \mathbf{e} = (R + j\omega L)\ \mathbf{I}$$

When used in this way the complex number $R + j\omega L$ is called an impedance operator in order to distinguish it from the magnitude of the impedance viz. $\sqrt{R^2 + \omega^2 L^2}$, the latter often being called simply the impedance. Some writers define $R + j\omega L$ as the vector impedance and $\sqrt{R^2 + \omega^2 L^2}$ as the scalar impedance. It must be remembered however that $R + j\omega L$ is not itself a vector but a vector operator. The preceding results are often conveniently thrown into another form in which the point hitherto denoted by 0 is regarded as the origin of a system of polar co-ordinates, the length and direction of the vector whose magnitude is r being denoted by r, θ. If a vector is represented by $(a + jb)\ (1)$ where (1) is a unit vector, its magnitude or size is $\sqrt{a^2 + b^2} = r$, where $a = r \cos \theta$, $b = r \sin \theta$,
and therefore

$$(a + jb)\ (1) = r(\cos \theta + j \sin \theta)\ (1)$$

hence the expression $r\ (\cos \theta + j \sin \theta)$ is equivalent to the operator $a + jb$. The factor r in this expression is equal to $\sqrt{a^2 + b^2}$ and is called the modulus of $a + jb$, while the angle θ which is equal to $tan^{-1} \frac{b}{a}$ is called the argument of the complex quantity.

59. Complex numbers can be multiplied together in the same way as ordinary numbers, thus $(a + jb)(c + jd) = ac + jad + jbc + j^2bd$, or (since $j^2 = -1$) $(a + jb)(c + jd) = (ac - bd) + j(ad + bc)$.

To divide one complex number by another an artifice is employed, e.g. $\dfrac{a + jb}{c + jd}$ is found by first expressing the denominator as a real quantity. To do this another complex number is introduced into the equation, namely $c - jd$, so that the quantity becomes

$$\frac{a + jb}{c + jd} \times \frac{c - jd}{c - jd}$$

$c - jd$ and $c + jd$ are called conjugate complex numbers. $(c + jd)(c - jd) = c^2 - jcd + jcd - j^2d$ or $c^2 + d^2$, because $-j^2 = 1$.

$$\text{Hence} \quad \frac{a + jb}{c + jd} = \frac{ac - jad + jbc - j^2bd}{c^2 + d^2}$$

$$= \frac{ac + bd}{c^2 + d^2} + j\frac{bc - ad}{c^2 + d^2}$$

$$= A + jB$$

which is a new complex number. The process of expressing the quotient of two complex quantities as a single complex quantity in this way is known as rationalisation.

Demoivres theorem

60. We have seen that if the operator $r(\cos\theta + j\sin\theta)$ operates upon a unit vector it has an effect equal to $a + jb$ if $a = r\cos\theta$ and $b = r\sin\theta$. Similar results are obtained if the operation is performed upon any vector, for example if

$$\mathbf{u} = (\cos\theta + j\sin\theta)\,\mathbf{v}$$

where $\mathbf{v}$ is a vector, and $\mathbf{u}$ is the result of operation upon it, the vector $\mathbf{u}$ is equal in magnitude to the vector $\mathbf{v}$ but is rotated in an anti-clockwise direction through the angle $\tan^{-1}\dfrac{b}{a}$. The inverse operation denoted by

$$\mathbf{v} = \frac{\mathbf{u}}{\cos\theta + j\sin\theta}$$

means that $\mathbf{v}$ is equal in magnitude to $\mathbf{u}$ but is rotated through the angle θ in a clockwise direction. Now, by the process of rationalisation

$$\frac{1}{\cos\theta + j\sin\theta} = \frac{\cos\theta - j\sin\theta}{\cos^2\theta + \sin^2\theta}$$

$$= \cos\theta - j\sin\theta, \text{ because } \cos^2\theta + \sin^2\theta = 1$$

As $\cos\theta = \cos(-\theta)$ and $-\sin\theta = \sin(-\theta)$, $\cos\theta + j\sin(-\theta)$ or $\cos\theta - j\sin\theta$ is an operator which causes clockwise rotation. Successive rotations, say of θ and ϕ, are equivalent to a single rotation $\theta + \phi$. That is to say the operation denoted by the expression

$\cos(\theta + \phi) + j\sin(\theta + \phi)$ is the effect of two successive rotations, the first being $(\cos\theta + j\sin\theta)$ and the second $(\cos\phi + j\sin\phi)$. Hence

$$\cos(\theta + \phi) + j\sin(\theta + \phi) = (\cos\theta + j\sin\theta)(\cos\phi + j\sin\phi)$$

If $\theta = \phi$

$$\cos 2\theta + j\sin 2\theta = (\cos\theta + j\sin\theta)^2$$

and if instead of only $\theta + \theta$, we have a sum of n rotations each equal to θ :—

$$\theta + \theta + \theta \ldots \text{etc. } n \text{ times} = n\theta$$

$\cos n\theta + j \sin n\theta = (\cos \theta + j \sin \theta)^n$. From this important theorem many useful results can be derived, for example the addition formulae of trigonometry which are frequently used in alternating current theory, particularly in the consideration of modulated waves :—

$$\cos (a + b) + j \sin (a + b) = (\cos a + j \sin a)\ (\cos b + j \sin b)$$
$$= \cos a \cos b + j \cos a \sin b + j \sin a \cos b - \sin a \sin b$$

Equating the real and imaginary parts

$$\cos (a + b) = \cos a \cos b - \sin a \sin b$$
$$\sin (a + b) = \cos a \sin b + \sin a \cos b$$

If instead of b, we write $- b$, in the original equation, we obtain

$$\cos (a - b) = \cos a \cos b + \sin a \sin b$$
$$\sin (a - b) = \sin a \cos b - \cos a \sin b$$

While if $a = b$,

$$\cos 2a + j \sin 2a = (\cos a + j \sin a)^2$$
$$= (\cos^2 a + 2j \sin a \cos a - \sin^2 a$$

Hence $\cos {}^2a \quad = \cos^2 a - \sin^2 a$

and $\sin 2a \quad = 2 \sin a \cos a$

adding $\sin {}^2a - \sin {}^2a$ (which is zero) to the right-hand side of the first of the two previous results

$$\cos 2a = \cos {}^2a + \sin {}^2a - 2 \sin {}^2 a$$
$$= 1 - 2 \sin {}^2 a$$

because $\cos^2 a + \sin^2 a = 1$

Rearranged this becomes

$$\sin^2 a = \tfrac{1}{2}\ (1 - \cos 2\ a).$$

a result we have already used in dealing with the average value of a sinusoidal curve over a complete period. Reverting to a former pair of expressions,

$$\cos (a + b) - \cos (a - b) = \cos a \cos b - \sin a \sin b - \cos a \cos b - \sin a \sin b$$

or

$$\cos (a - b) - \cos (a + b) = 2 \sin a \sin b$$

As a practical example of the use of these formulae, consider the expression

$$i = \mathcal{I}\ (1 + \sin \omega_a t) \sin \omega_r t,$$

where ω_a is 2π times an audio-frequency f_a and ω_r is 2π times a radio-frequency f_r. In Chapter XII it is shown that the aerial current in an R/T transmitter may be of this form.

Expanding the right-hand member

$$i = \mathcal{I}\ [\sin \omega_r t + \sin \omega_r t \sin \omega_a t]$$
$$= \mathcal{I}\ [\sin \omega_r t + \tfrac{1}{2} \{\cos (\omega_r - \omega_a)t - \cos (\omega_r + \omega_a)t\}]$$

so that the complicated expression first given may be resolved into the sum of three sinusoidal waves of different frequencies.

61. In order to develop still another method of representation of a vector quantity by algebraic symbols, we must first refer to what are termed algebraic series. The latter expression is used to denote a number of terms each of which is related to its predecessor and successor in a perfectly definite manner. For instance, 1, 2, 3, 4, 5, is a series, each term being formed by adding unity to the preceding one, while 1, 2, 4, 8, etc. is a series in which each term is formed by multiplying the preceding one by two. An example of the development of a series is found in

the algebraic expression $(a + x)^n$, which gives rise to what is known as the binomial series. The binomial theorem states that, no matter what values a, x and n possess, i.e., positive or negative, integral or fractional,

$$(a + x)^n = a^n + \frac{n}{1} \cdot a^{n-1} \cdot x + \frac{n}{1} \cdot \frac{n-1}{2} \cdot a^{n-2} \cdot x^2$$

$$+ \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot a^{n-3} \cdot x^3 \dots\dots\dots$$

This is easily proved when n is a small positive integer, e.g. 2, 3, or 4, by direct multiplication;

$$a + x)^2 = a^2 + \frac{2}{1} \cdot a^{2-1}x + \frac{2}{1} \cdot \frac{2-1}{2} \cdot a^{2-2} \cdot x^2$$

$$+ \frac{2}{1} \cdot \frac{2-1}{2} \cdot \frac{2-2}{3} \cdot a^{2-3} \cdot x^3 \dots\dots\dots$$

$$= a^2 + 2a^1x + \frac{2}{1} \cdot \frac{1}{2} \cdot a^0 \cdot x^2$$

$$+ 2 \cdot \frac{1}{2} \cdot \frac{0}{3} \cdot a^{-1} \cdot x^3 \dots\dots\dots$$

$$= a^2 + 2ax + x^2$$

because $a^0 = 1$ and the fourth term of the series and all subsequent terms contain 0 as a factor and are therefore equal to zero.

Similarly

$$(a + x)^3 = a^3 + 3a^2x + 3ax^2 + x^3$$

The arrangement of the terms can be seen to follow a definite plan. Starting with a^n, the powers of a decrease by unity in each successive term until (if n is a positive integer) a^0 or unity is reached, when the series ceases. This, however, does not occur if n is fractional or negative, for no term ever contains a^0, consequently in such instances the series continues indefinitely. The powers of x increase as the powers of a diminish, the first term being $a^n x^0$ or a^n, the second containing x^1, the third x^2 and so on. The numerical coefficients can easily be written, provided n is a positive integer, by the use of the following table.

1	
1 1	
1 2 1	$1a^2 + 2ab + 1b^2$
1 3 3 1	$1a^3 + 3a^2b + 3ab^2 + 1b^3$
1 4 6 4 1	$1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4$ etc.
1 5 10 10 5 1	
1 6 15 20 15 6 1	
1 7 21 35 35 21 7 1	

It will be observed that any integer in this triangle is the sum of the two adjacent to it in the line above, except at the ends of each line where the integer is always unity. By adding the next line in the table for instance we may readily obtain the expansion of $(a + x)^8$

$$(a + x)^8 = a^8 + 8a^7x + 28a^6x^2 + 56a^5x^3 + 70a^4x^4 + 56a^3x^5 + 28a^2x^6 + 7ax^7 + x^8$$

62. Let the expression $\left(1 + \frac{1}{n}\right)^n$ be expanded by the binomial theorem. The series obtained is

$$1^n + \frac{n}{1} \cdot 1^{n-1} \cdot \frac{1}{n} + \frac{n}{1} \cdot \frac{n-1}{2} \cdot 1^{n-2} \cdot \left(\frac{1}{n}\right)^2$$

$$+ \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot 1^{n-3} \cdot \left(\frac{1}{n}\right)^3 \ldots\ldots \text{etc.}$$

$$= 1 + 1 + \frac{n(n-1)}{1 \times 2n^2} + \frac{n(n-1)(n-2)}{1 \times 2 \times 3n^3} + \frac{n(n-1)(n-2)(n-3)}{1 \times 2 \times 3 \times 4n^4} \ldots\ldots$$

$$= 1 + 1 + \frac{1 - \frac{1}{n}}{2} + \frac{\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)}{2 \times 3} + \frac{\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\left(1 - \frac{3}{n}\right)}{2 \times 3 \times 4} \ldots\ldots\ldots$$

Now suppose n to become larger and larger until it is some quantity greater than is comprehensible, which we may denote by the sign $\propto$. All the fractions of the form $\frac{2}{n}$, $\frac{3}{n}$ etc., become utterly insignificant when added to or subtracted from unity, and it may then be stated that the limiting value of $\left(1 + \frac{1}{n}\right)^n$, when $n \rightarrow \propto$ (or when $\frac{1}{n} \rightarrow 0$) is

$$1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} \ldots\ldots\ldots \text{etc.}$$

where $2! = 1 \times 2$, $3! = 1 \times 2 \times 3$ etc., $3!$ is called "factorial 3", $n!$ "factorial n" and so on. The sign $\rightarrow$ is read "approaches the limiting value".

The sum of an infinitely large number of terms of this series is 2·718281828.... which is denoted by the greek letter ε. The value of "ε" to five correct decimal places can be obtained by taking only ten terms of the series. This awkward-looking number is of great importance in physics, being connected with all natural processes of growth or decay, for instance the voltage to which a condenser is charged by an applied steady voltage E is

$$e_t = E\left(1 - \frac{1}{\varepsilon^{\frac{t}{CR}}}\right)$$

which is more conveniently written

$$e_t = E\left(1 - \varepsilon^{-\frac{t}{CR}}\right)$$

where e_t is the voltage t seconds after the charge commences, C the capacitance of the condenser, R the total resistance in series with it, and E the applied E.M.F.

Now suppose that the expression to be expanded, (that is, to be expressed as the sum of a series of terms) is $\left(1 + \frac{1}{n}\right)^{nx}$ This becomes, by employment of the binomial theorem

$$\left(1 + \frac{1}{n}\right)^{nx} = 1 + \frac{nx}{n} + \frac{nx(nx-1)}{2!\,n^2}$$

$$+ \frac{nx(nx-1)(nx-2)}{3!\,n^3} \ldots\ldots\ldots \text{etc.}$$

and by sufficiently increasing n, as in the previous discussion, the terms $nx - 1$, $nx - 2$ etc., can be made to differ inappreciably from nx. When $n \to \infty$, therefore this series becomes

$$\left\{\left(1 + \frac{1}{n}\right)^n\right\}^x = \varepsilon^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \ldots\ldots\ldots \text{etc.}$$

In the above example

$$\varepsilon^{\frac{t}{CR}} = 1 + \frac{t}{CR} + \frac{\left(\frac{t}{CR}\right)^2}{2!} + \frac{\left(\frac{t}{CR}\right)^3}{3!} \ldots\ldots\ldots\ldots \text{etc.}$$

If x is a negative quantity, it can be shown in the same way that

$$\varepsilon^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} \ldots\ldots\ldots\ldots \text{etc.}$$

Thus

$$\varepsilon^{-\frac{t}{CR}} = 1 - \frac{t}{CR} + \frac{\left(\frac{t}{CR}\right)^2}{2!} - \frac{\left(\frac{t}{CR}\right)^3}{3!} \ldots\ldots\ldots\ldots \text{etc.}$$

Exponential form of $\cos\theta + j\sin\theta$.

63 As n rotations through an angle of θ radians are equal to a single rotation of $n\theta$ radians

$$\cos n\theta + j\sin n\theta = (\cos\theta + j\sin\theta)^n$$

$$= \cos^n\theta\left(1 + j\frac{\sin\theta}{\cos\theta}\right)^n$$

$$= \cos^n\theta\,(1 + j\tan\theta)^n$$

If $n\theta = \varphi$

$$\cos\varphi + j\sin\varphi = \cos^n\frac{\varphi}{n}\left(1 + j\tan\frac{\varphi}{n}\right)^n,$$

and if n is allowed to become larger and larger without limit $\tan\frac{\varphi}{n}$ and $\frac{\varphi}{n}$ become more nearly equal to each other. In the limit, $\tan\frac{\varphi}{n} = \frac{\varphi}{n}$ and

$$\cos\varphi + j\sin\varphi = \cos^n\frac{\varphi}{n}\left(1 + j\frac{\varphi}{n}\right)^n$$

or, since the term $\cos^n\frac{\varphi}{n}$ also approaches unity as n is increased without limit

$$\cos\varphi + j\sin\varphi = \left(1 + j\frac{\varphi}{n}\right)^n \text{ if } n \to \infty.$$

The right hand member of this equation may be expanded by the binomial theorem, with the following result;

$$\cos\varphi + j\sin\varphi = 1 + j\varphi + \frac{(j\varphi)^2}{2!} + \frac{(j\varphi)^3}{3!} + \frac{(j\varphi)^4}{4!} \ldots \text{etc.}$$

Comparing the righthand member with the expansion of ε^x it will be seen that they are of the same form, and therefore the operator $(\cos\varphi + j\sin\varphi)$ is written in the alternative form $\varepsilon^{j\phi}$. It can also be shown that $\cos\varphi - j\sin\varphi$ is formally equivalent to $\varepsilon^{-j\phi}$.

It has already been shown that

$$(\cos\theta + j\sin\theta)(\cos\varphi + j\sin\varphi) = \cos(\theta+\varphi) + j\sin(\theta+\varphi)$$

and in the exponential form this becomes $\varepsilon^{j\theta}\varepsilon^{j\phi} = \varepsilon^{j(\theta+\phi)}$. The imaginary index therefore enters into algebraic combination just as if it were a real number.

The four methods of expressing a vector operator are equivalent to each other, and if $Z \underline{/\theta}$, $a + jb$, $Z(\cos\theta + j\sin\theta)$, $\varepsilon^{j\theta}$ are applied in turn to a unit vector, the effect of the operation is in every case to rotate the vector in the positive direction through the angle θ and to extend the magnitude of the vector to Z units, provided that $a^2 + b^2 = Z^2$ and $\theta = \tan^{-1}\frac{b}{a}$

Example 12.—$3 + j\,4$ is the $a + j\,b$ form of a vector operator. Express in the other forms.

The modulus of $3 + j\,4$ is $\sqrt{3^2 + 4^2} = \sqrt{25} = 5$

The argument, θ, is $\tan^{-1}\frac{4}{3} = \cdot 927$ radians approx.

$$3 + j\,4 = 5(\cos\cdot 927 + j\sin\cdot 927)$$
$$\cos\cdot 927 = \cdot 6$$
$$\sin\cdot 927 = \cdot 8$$
$$3 + j\,4 = 5(\cdot 6 + j\,\cdot 8)$$
$$\text{or } 5\,\underline{/\cdot 927}$$
$$\text{or } 5\,\underline{/53^\circ\,7'}$$
$$\text{or } 5\,\varepsilon^{(\cdot 927)}$$

The advantage of the form $\varepsilon^{j\phi}$ is the manner in which it lends itself to multiplication and division of operators. This process is simply carried out as follows :—

$$\text{if } a + jb = Z_1\,\varepsilon^{j\phi}$$
$$c + jd = Z_2\,\varepsilon^{j\theta}$$
$$(a + jb)(c + jd) = Z_1 Z_2\,\varepsilon^{j(\theta-\phi)}$$

While $$Z_1\,\varepsilon^{j\phi} \div Z_2\,\varepsilon^{j\theta} \text{ is } \frac{Z_1}{Z_2}\,\varepsilon^{j(\phi-\theta)}$$

The exponential form $Z\,\varepsilon^{j\phi}$ is therefore the most convenient when multiplication or its extensions are to be performed, while the form $a + jb$ is preferable when addition or subtraction of vector operators is contemplated. For this reason it is often advisable to change from one form to the other in the course of an analysis.

64. As an example of the simplification introduced into A.C. calculations by the employment of the impedance operator, the selectivity of the circuit consisting of an inductance and capacitance in parallel will now receive further consideration. In the first place, it must be emphasised that this parallel combination (which for the sake of brevity may be referred to as a " rejector circuit ", even though the particular application is not that of a true rejector), is only of practical utility when used in combination with other circuits, which may possess only resistance, or may be tuned to any frequency whatever. In many instances the object of employing the rejector circuit is to obtain maximum P.D. across its ends at a certain frequency (to which the rejector is tuned) while the P.D. set up by currents of other frequencies is required to be a minimum. Such a circuit is shown in fig. 44 in which e_r represents a source of alternating E.M.F. of frequency f_r and e_n a source of alternating E.M.F. having an amplitude equal to that of e_r but with a frequency variable from 0 to ∞, the R.M.S. values being E_r, E_n. In series with the rejector circuit is a fixed resistance r, and it is desired to obtain as large a P.D. as possible

between the points A and B, at the frequency f_r, to which the rejector is tuned, while at any other frequency f_n the P.D. between A and B is to be as low as possible. The selectivity of this combination may then be defined as the ratio

$$\frac{\text{P.D between A and B at the frequency } f_r}{\text{P.D between A and B at any other frequency}} = \frac{V_r}{V_n}$$

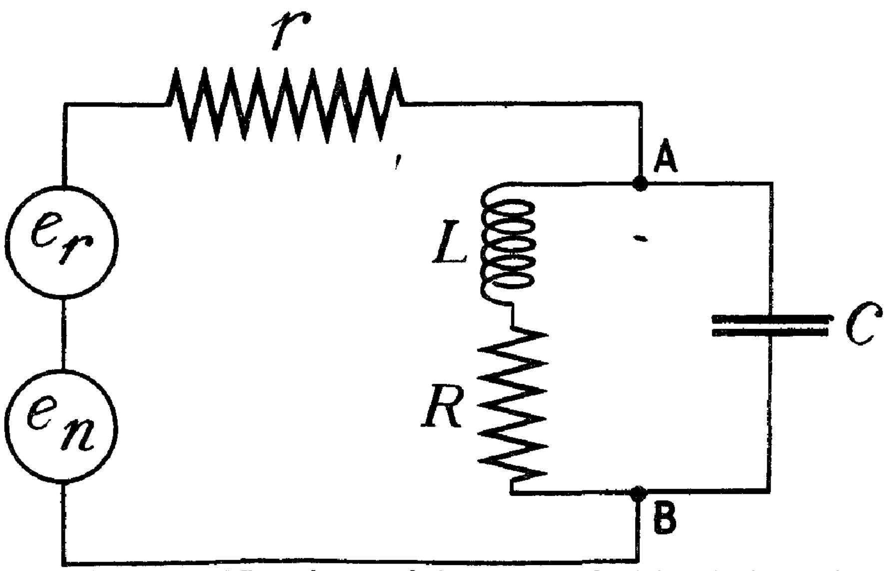


Fig. 44, CHAP. V.—Effect of external circuit upon selectivity of rejector circuit.

Referring to the diagram, the impedance operator, z, of the rejector circuit, at any irequency $\frac{\omega}{2\pi}$ is found by the rule for parallel impedances, which is identical with that for parallel resistances, provided the impedances are expressed in the form of vector operators. Denoting the impedance operator of the capacitive branch by z_c and that of the inductive branch by z_L, $z_c = \frac{1}{j\omega C}$ and $z_L = R + j\omega L$,

$$z = \frac{z_c\, z_L}{z_c + z_L}$$

$$= \frac{\frac{1}{j\omega C}(R + j\omega L)}{R + j\omega L + \frac{1}{j\omega C}}$$

$$= \frac{R + j\omega L}{j\omega CR + 1 - \omega^2 LC}$$

The current $\mathbf{i}$, due to a sinusoidal E.M.F. $\mathbf{e}$, is

$$\mathbf{i} = \frac{\mathbf{e}}{r + z}$$

and the P.D. between A and B is $\mathbf{v} = z\,\mathbf{i}$ or

$$\mathbf{v} = \frac{z\,\mathbf{e}}{r + z}$$

Denoting the impedance operator of the rejector at the frequency f_r by z_r and the impedance operator at any other frequency f_n by z_n

$$\mathbf{v}_r = \frac{z_r\,\mathbf{e}}{r + z_r}$$

$$\mathbf{v}_r = \frac{z_n\,\mathbf{e}}{r + z_n}$$

Since the amplitudes of e_r and e_n are numerically equal, however, the ratio $\frac{\mathbf{v}_r}{\mathbf{v}_n}$

$$\frac{\mathbf{v}_r}{\mathbf{v}_n} = \frac{z_r}{r + z_r} \div \frac{z_n}{r + z_n}$$

Inserting the values of z_r and z_n and observing that at the frequency f_r, $1 - \omega_r^2 LC = 0$,

$$\frac{\mathbf{v}_r}{\mathbf{v}_n} = \frac{\dfrac{R + j\omega_r L}{j\omega_r CR}}{r + \dfrac{R + j\omega_r L}{j\omega_r CR}} \times \frac{r + \dfrac{R + j\omega_n L}{j\omega_n CR + 1 - \omega_n^2 LC}}{\dfrac{R + j\omega_n L}{j\omega_n CR + 1 - \omega_n^2 LC}}$$

In the practical application of a circuit of this type R is always small compared with ωL and $R + j\omega L$ may be replaced without appreciable error by $j\omega L$.

$$\frac{\mathbf{v}_r}{\mathbf{v}_n} = \frac{j\omega_r L}{j\omega_r\,(CRr + L)} \times \frac{r\,(j\omega_n CR + 1 - \omega_n^2 LC) + j\omega_n L}{j\omega_n L}$$

$$= \frac{\omega_r}{\omega_n}\left\{\frac{j\omega_n\,(CRr + L) + r\,(1 - \omega_n^2\,LC)}{j\omega_r\,(CRr + L)}\right\}$$

or, since $LC = \dfrac{1}{\omega_r^2}$

$$\frac{\mathbf{v}_r}{\mathbf{v}_r} = \frac{\omega_r}{\omega_n}\left\{\frac{j\omega_n\,(CRr + L) + r\left(1 - \dfrac{\omega_n^2}{\omega_r^2}\right)}{j\omega_r\,(CRr + L)}\right\}$$

$$= \frac{\omega_r}{\omega_n}\left\{\frac{\omega_n}{\omega_r} + \frac{r\left(1 - \dfrac{\omega_n^2}{\omega_r^2}\right)}{j\omega_r\,(CRr + L)}\right\}$$

$$= 1 + \frac{r\left(1 - \dfrac{\omega_n^2}{\omega_r^2}\right)}{j\,\omega_n\,(CRr + L)}$$

The second term of the right hand member of the equation may be further simplified :—

$$\frac{r\left(1 - \dfrac{\omega_n^2}{\omega_r^2}\right)}{j\omega_n\,(CRr + L)} = \frac{r\,(\omega_r^2 - \omega_n^2)}{j\omega_n\omega_r^2\,(CRr + L)} = \frac{\omega_r^2 - \omega_n^2}{j\omega_n\left(\dfrac{R}{L} + \dfrac{1}{Cr}\right)}$$

hence

$$\frac{\mathbf{v}_r}{\mathbf{v}_n} = 1 - j\,\frac{\omega_r^2 - \omega_n^2}{\omega_n\left(\dfrac{R}{L} + \dfrac{1}{C_r}\right)}$$

In this form it may be seen that in order to obtain high selectivity the factor $\frac{R}{L}+\frac{1}{Cr}$ must be as small as possible. Thus in some practical applications of the rejector circuit selectivity is not necessarily achieved by making $\frac{L}{R}$ large, for the product LC is fixed by the value of the " desired " frequency that is by ω_r, hence a large value of L entails a small value of C, and consequently a large value of $\frac{1}{Cr}$. The effect will now be illustrated by a numerical example.

Example 13.—If $L = 160\ \mu H$, $C = \cdot 00025\ \mu F$, $R = 50$ ohms, $r = 5{,}000$ ohms, find the selectivity when $\omega_n = 5\cdot 6 \times 10^6$.

Considering the imaginary portion of (ii) only

$$\omega_r = \sqrt{\frac{1}{LC}} = \sqrt{\frac{10^6}{160} \times \frac{10^6}{\cdot 00025}} = 5 \times 10^6$$

$$\begin{aligned}\omega_r^2 - \omega_n^2 &= (\omega_r + \omega_n)(\omega_r - \omega_n) = 10\cdot 6 \times 10^6 \times \cdot 6 \times 10^6 \\ &= 6\cdot 36 \times 10^{12}\end{aligned}$$

$$\frac{\omega_r^2 - \omega_n^2}{\omega_n} = \frac{6\cdot 36 \times 10^{12}}{5\cdot 6 \times 10^6} = 1\cdot 135 \times 10^6$$

$$\begin{aligned}\frac{R}{L} + \frac{1}{Cr} &= \frac{50}{160} \times 10^6 + \frac{10^6}{\cdot 00025 \times 5{,}000} \\ &= \cdot 3125 \times 10^6 + \cdot 8 \times 10^6. \\ &= 1\cdot 1125 \times 10^6.\end{aligned}$$

$$\begin{aligned}\frac{V_r}{V_n} &= 1 - j\,\frac{1\cdot 135 \times 10^6}{1\cdot 1125 \times 10^6} \\ &= 1 - j\,1\cdot 011.\end{aligned}$$

In R.M.S. values, therefore

$$V_r = \sqrt{1^2 + (1\cdot 011)^2}\ V_n$$

$$\text{or } \frac{V_r}{V_n} \doteqdot 1\cdot 42.$$

Now suppose that r is increased to 100,000 ohms. Then $\frac{1}{Cr}$ becomes 4×10^4 which is small compared with $\frac{R}{L}$, and the latter factor will control the selectivity, giving

$$\begin{aligned}S &= 1 - j\,\frac{1\cdot 135 \times 10^6}{(\cdot 3125 + \cdot 04) \times 10^6} \\ &= 1 - j\,\frac{1\cdot 135}{\cdot 3525} \\ &= 1 - j\,3\cdot 22\end{aligned}$$

$$\text{That is, } V_r = \sqrt{1 + (3\cdot 22)^2}\ V_n$$

$$\text{or } \frac{V_r}{V_n} = 3\cdot 375.$$

The voltage between A and B (fig. 44) due to the non-resonant frequency is in this instance only 29·6 per cent. of the voltage at resonance, for equal applied voltages.

If the value of the ratio $\frac{L}{C}$ is chosen so that it is equal to Rr or in other words if $\frac{L}{CR} = r$ the factor $\left(\frac{R}{L} + \frac{1}{Cr}\right)$ has its minimum value. This may be proved mathematically or by taking simple numerical values. It follows that if in fig. 44 r represents the total internal resistance of a circuit which is equivalent in its effect to the generators e_r, e_n, specified above, the greatest selectivity will be obtained when the dynamic resistance of the rejector circuit is equal to the internal resistance of the equivalent generator. This is of the utmost importance in the design of the circuits used in connection with thermionic valves both for reception and transmission, and the above example shows the danger of making any assumption as to the selectivity of a rejector circuit without taking into account the circuits to which the rejector is connected.

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